Unit 5: Sequences, Series, and Patterns

Section 1: Sequences and Series

1. Sequence: an ordered list of numerical terms
2. Finite Sequence: has a first term (a beginning) and a last term (an end)
3. Infinite Sequence: a sequence with either no beginning or no end, or neither beginning nor end

$$\boldsymbol{Finite\ vs.\ Infinite\ Sequence}$$

$\boldsymbol{Examples\ of\ Finite\ Sequences}$

- $\{1, 2, 3, 4, 5\}$
- $\{1, 2, 3, \dots, n\}, where\ n < \infty$
- $\{a, \dots, -2, -1, 0, 1, 2, \dots b\}, where\ -\infty < a < b < \infty$

$\boldsymbol{Examples\ of\ Infinite\ Sequences}$

- $\{1, 2, 3, \dots\}$
- $\{\dots, -2, -1, 0, 1, 2, 3\}$
- $\{\dots, -2, -1, 0, 1, 2, \dots\}$

4. Explicit Formula: defines the n^{th} (general) term. Each term can be generated by substituting a counting number for n.

$$\boldsymbol{Example\ of\ an\ Explicit\ Formula}$$

$\boldsymbol{Given: t_n = 2n + 3.\ Then, for\ n = \{0, 1, 2, 3, \dots\}, t = \{3, 5, 7, 9, \dots\}}$
$\boldsymbol{This\ would\ allow\ you\ to\ find\ any\ term, say\ the\ 6^{th}\ term.\ t_6 = 2(6) + 3 = 15}$

5. Recursive Formula: one or more previous terms are used to generate the next term.

$$\boldsymbol{Example\ of\ a\ Recursive\ Formula}$$

$\boldsymbol{Given: t_n = 3t_{n-1}, and\ t_0 = 4.\ Then, t_1 = 12, t_2 = 36, t_3 = 108, \dots}$

A German astronomer defined a recursive sequence (illustrated below) for calculating the distances from each planet to the sun, based on the distance from the earth to the sun as 1 unit.

$$\boldsymbol{Bode's\ Sequence}$$

$$t_1 = \frac{(0+4)}{10} = 0.4, t_2 = \frac{(3+4)}{10} = 0.7, t_3 = \frac{(6+4)}{10} = 1, \dots, t_8 = \frac{(192+4)}{10} = 19.6$$

$\boldsymbol{This\ is\ defined\ recursively\ as\ t_1 = 0.4, t_2 = 0.7, t_n = 2t_{n-1} - 0.4,}$

$\boldsymbol{for\ n = \{3, 4, 5, 6, 7, 8\}.\ Note, t_3 = Earth \leftrightarrow Sun = 1\ Bode\ unit.}$

Another, more famous, recursive sequence is the Fibonacci sequence (illustrated below). The Fibonacci is also known as the "golden spiral," because it often occurs in nature. For example, the seeds on a sun flower have a "golden spiral" pattern. The seeds are arranged in clockwise and counterclockwise spirals. The number of seeds in successive clockwise and counterclockwise spirals form a Fibonacci Sequence, meaning that number of seeds in each spiral (after the second) is equal to the sum of the number of seeds in the previous two spirals.

$$\boldsymbol{Fibonacci\ Sequence\ (called\ the\ golden\ spiral)}$$

$$\boldsymbol{t_1 = 1, t_2 = 1, t_3 = 2, t_4 = 3, t_5 = 5, t_6 = 8, t_7 = 13, \ldots This\ is\ defined}$$

$$\boldsymbol{recursively\ as\ t_1 = 1, t_2 = 1, t_n = t_{n-2} + t_{n-1}, for\ n = \{3, 4, 5, \ldots\}}$$

6. Series: an expression that indicates the sum of terms of a sequence.
   $Example: 1 + 2 + 3 + 4 + 5 + 6 + \cdots$
7. Summation Notation: uses the Greek letter sigma, Σ, to express a series in abbreviated form. The initial value of n is indicated beneath the symbol, the terminal value of n, is indicated above the symbol, and the explicit formula is indicated to the right.

$$\boldsymbol{Example\ of\ a\ Series\ in\ Summation\ Notation}$$

$$\sum_{k=1}^{6}(3k-1) = 2 + 5 + 8 + 11 + 14 + 17 = 57$$

$$\boldsymbol{TI-84\ Plus: 2nd, LIST, MATH, sum(2nd, LIST, OPS, seq(3x-1, x, 1, 6, 1)) = 57}$$

$$\boldsymbol{Summation\ Properties}:$$

$$\boldsymbol{For\ sequences\ a_k\ and\ b_k\ and\ natural\ number\ (positive\ integer) n}:$$

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k\ , and\ \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\boldsymbol{Sumation\ Formulas\ for\ Constant, Linear, and\ Quadratic\ Series}:$$

$$\boldsymbol{The\ following\ hold\ true\ for\ all\ natural\ numbers\ \{n = 1, 2, 3, 4, \ldots\}}:$$

Constant	***Linear***	***Quadratic***
$\sum_{k=1}^{n} c = nc$	$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Assignment: Holt Algebra 2, Pages 696-697 (10-56 even, 76, 82)

Section 2: Arithmetic Sequences

1. Arithmetic Sequence: a sequence of two or more terms in which the difference between successive (consecutive) terms is the same number, d, called the common difference.

Formula For d:

$$d = t_n - t_{n-1}$$

Formula for the nth Term of an Arithmetic Sequence:

$$t_n = t_1 + (n-1)d$$

2. Arithmetic Means of Two Numbers: the terms between the two numbers, which with the two numbers, form an arithmetic sequence.

Formula for k Arithmetic Means between a and b:

$$d = \frac{(b-a)}{(k+1)}, \text{ and } m_1 = a + d, m_2 = a + 2(d), \ldots, m_k = a + (k)(d) = b - d$$

Assignment: Holt Algebra 2, Pages 703-705 (8-82 even)

Section 3: Arithmetic Series

1. Arithmetic Series: the indicated sum of the terms of an arithmetic sequence

Formula for the Sum of the First n terms of an Arithmetic Series:

$$\sum_{k=1}^{n} t_k = S_n = n\left(\frac{t_1 + t_n}{2}\right) = n\left(\frac{2t_1 + (n-1)d}{2}\right)$$

Assignment: Holt Algebra 2, Pages 710-712 (8-54 even)

Section 4: Geometric Sequences

1. Geometric Sequence: a sequence of two or more nonzero terms in which the ratio of successive (consecutive) terms is the same number, r, called the common ratio.

Formulas for r:

$$r = \frac{t_n}{t_{n-1}}, \qquad r = \sqrt[(n-1)]{\left(\frac{t_n}{t_1}\right)}$$

Formula for the nth Term of a Geometric Sequence:

$$t_n = t_1 r^{(n-1)}$$

2. Geometric Means of Two Numbers: the terms between the two numbers, which with the two numbers, form a geometric sequence.

Formula for k Geometric Means between terms a and b, where $a \neq 0$:

$$r = \sqrt[(n-1)]{\frac{b}{a}} = \sqrt[(k+1)]{\frac{b}{a}}$$

$$g_1 = ar, g_2 = ar^2, g_3 = ar^3, \dots, g_k = ar^k = \frac{b}{r}$$

Assignment: Holt Algebra 2, Pages 717-719 (8-74 even, 75-78 all)

Section 5: Geometric Series and Mathematical Induction

1. Geometric Series: the indicated sum of the terms of a geometric sequence.

$$\textit{Formula for the Sum of the First n Terms of a Geometric Series}:$$

$$S_n = t_1\left(\frac{(1-r^n)}{(1-r)}\right), \textit{where } r \neq 1.$$

2. Mathematical Induction: To prove that a statement is true for all numbers n:

Steps for Proving a Statement by Induction

***Basic Step**: Show it is true for $n = 1$*

***Induction Step**: Assume it is true for a natural number k,*

and prove this assumption implies it is true for $k + 1$.

Note: Proofs in geometry use deductive reasoning (a logical chain justified by theorems, definitions, properties, postulates …). Inductive reasoning is based on showing that a pattern must remain consistent through its entirety.

$$\textit{Example: Prove}: (1+2+3+\cdots+n) = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$\textit{For } n = 1: \sum_{j=1}^{1} j = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$$

$$\textit{Assume}: (1+2+3+\cdots+k) = \sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

$$\textit{Then}: \big(1+2+3+\cdots+k+(k+1)\big) = \sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1)$$

$$= \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)}{2} + \frac{2k+2}{2} = \frac{k^2+k+2k+2}{2}$$

$$= \frac{k^2+3k+2}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)\big((k+1)+1\big)}{2}$$

$$\textit{Therefore } (\therefore): (1+2+3+\cdots+n) = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Assignment: Holt Algebra 2, Pages 724-726 (10-72 even)

Section 6: Infinite Geometric Series

1. Infinite Geometric Series: a geometric series with infinitely many terms.
2. Partial Sum of an Infinite Series: the sum of a given number of terms and not the sum of the entire series. ($S_k = the\ sum\ of\ the\ first\ k\ terms\ of\ a\ series.$)
   For example, $S_1 = (t_1);\ S_2 = (t_1 + t_2);\ S_3 = (t_1 + t_2 + t_3), \ldots; S_n = (t_1 + t_2 + t_3 + \cdots + t_n)$
3. Convergence: when partial sums of an infinite series approach a fixed number as n increases.
4. Divergence: when partial sums of an infinite series do not approach a fixed number as n increases.

Examples of Convergent and Divergent Geometric Series:

Convergent	***Divergent***
$\boldsymbol{S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \left(\frac{1}{2}\right)^n}$ $\boldsymbol{S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}, S_4 = \frac{15}{16}, \ldots}$ ***The partial sums appear to approach 1.***	$\boldsymbol{S_n = -2 + 4 - 8 + \cdots + (-2)^n}$ $\boldsymbol{S_1 = -2, S_2 = 2, S_3 = -6, S_4 = 10, \ldots}$ ***The partial sums appear to be alternating between a larger negative and a larger positive number, not a fixed number.***

Formula for the Sum of a Convergent Infinite Geometric Series:

If a geometric sequence has a common ratio r and $|r| < 1$, then the sum, S, of the related infinite geometric series is as follows:

$$S_n = \frac{t_1}{(1 - r)}$$

This formula yields $S_n = 1$ for the convergent example above.

Assignment: Holt Algebra 2, Pages 732-734 (8-64 even)

Section 7: Pascal's Triangle

1. Pascal's Triangle: a pattern in the form of a triangle, where each row begins and ends with "1," and each of the interior terms is the sum of the two terms directly above it. The top row is called row 0, because anything to the 0 power is 1 ($i.e.\ (a+b)^0 = 1$).

Patterns in Pascal's Triangle:

- $\textit{Row } n \textit{ contains } n+1 \textit{ terms}$
- $\textit{The kth term in row } n \textit{ is } \binom{n}{k-1} = {}_nC_{k-1} = \frac{n!}{(k-1)!(n-(k-1))!} = \frac{n!}{(k-1)!(n-k+1)!}$
- $\textit{The sum of all terms in row } n \textit{ is } 2^n$
- ${}_nC_{k-1} + {}_nC_k = {}_{n+1}C_k = \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, \textit{where } 0 < k \le n$

$$
\begin{array}{lcccccccc}
\text{Row } 0 \rightarrow & 1 & & & & & & \leftrightarrow & {}_0C_0 \\
\text{Row } 1 \rightarrow & 1 & \boxed{1} & & & & & \leftrightarrow & {}_1C_0 \quad {}_1C_1 \\
\text{Row } 2 \rightarrow & 1 & \boxed{2} & 1 & & & & \leftrightarrow & {}_2C_0 \quad {}_2C_1 \quad {}_2C_2 \\
\text{Row } 3 \rightarrow & 1 & \boxed{3} & 3 & 1 & & & \leftrightarrow & {}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 \\
\text{Row } 4 \rightarrow & 1 & \boxed{4} & 6 & 4 & 1 & & \leftrightarrow & {}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4 \\
\text{Row } 5 \rightarrow & 1 & \boxed{5} & 10 & 10 & 5 & 1 & \leftrightarrow & {}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5 \\
 & \vdots & & & & & & & \vdots
\end{array}
$$

Pascal's Triangle and Two − Outcome Experiments with Equal Probability of Success or Failure:

If a probability experiment is repeated in n independent trials, and the probability of success is $P(S) =$ *the probability of failure* $= P(F)$, *then the probability* $P(A)$ *of event A occurring exactly k times is given by* $P(A) = \frac{{}_nC_k}{2^n}$.

Example: If a fair coin is flipped 10 times, (a) what is the probability that heads will appear exactly 7 times? (b) What is the probability that heads will appear at least 7 times? Solution (a): $P(7H) = \frac{{}_{10}C_7}{2^{10}} = 12\%$.

Solution (b): $P(7H, 8H, 9H, \text{or } 10H) = \frac{{}_{10}C_7}{2^{10}} + \frac{{}_{10}C_8}{2^{10}} + \frac{{}_{10}C_9}{2^{10}} + \frac{{}_{10}C_{10}}{2^{10}} = 17\%$

$$\boldsymbol{Pascal's\ Triangle\ and\ Two-Outcome\ Experiments}$$
$$\boldsymbol{with\ Different\ Probability\ of\ Success\ or\ Failure:}$$

If a probability experiment is repeated in n independent trials, and the probability of success is $P(S)$, and the probability of failure is $P(F) = 1 - P(S)$, then the probability $P(A)$ of event A occurring exactly k times is given by the formula $P(A) = \left(P(S)\right)^k \left(P(F)\right)^{(n-k)} ({}_nC_k)$.

Example: If a basketball player has a 72% free throw percentage, what is the probability that she will hit exactly 5 free throws out of 9 in a given game?
Solution: $P(S) = .72, P(F) = 1 - .72 = .28$

$$P(hit\ 5\ out\ of\ 9) = (.72)^5(.28)^4({}_9C_5) \approx .1498 \approx 15\%$$

Assignment: Holt Algebra 2, Pages 739-741 (8-26 even, 30-46 even)

Section 8: The Binomial Theorem

1. The Binomial Theorem: a theorem, derived from Pascal's Triangle, that is used to expand a binomial raised to a given power and write it in polynomial form.

$$\boldsymbol{The\ Binomial\ Theorem:}$$

Let n be any natural number (a positive integer).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} y^k.\ \ Recall, \binom{n}{k} = {}_nC_k, where\ k \le n.$$

$$= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{(n-1)} y^1 + \cdots + \binom{n}{n-1} x^1 y^{(n-1)} + \binom{n}{n} x^0 y^n$$

2. Using Pascal's Triangle to Expand Binomials: The coefficients of a binomial raised to the n^{th} power are equal to the terms in the n^{th} row in Pascal's Triangle. The exponents of the first term in the binomial begin with n and descend to 0. The exponents of the second term begin at 0 and ascend to n, as indicated in the Binomial Theorem, above.

$$\textbf{Examples of Binomial Expansion with Pascal's Triangle:}$$

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^4$$

Algorithm for Expanding Binomials with Compound Terms $(ax^j + by^k)^n$:

- ***Let*** $p = (ax^j)$ ***and*** $q = (by^k)$
- ***Expand*** $(p+q)^n$ ***using the Binomial Theorem or Pascal's Triangle.***
- ***Substitute*** (ax^j) ***for*** p, (by^k) ***for*** q, ***and simplify***

Example: ***Expand*** $(2x - 5y^2)^3$

- ***Let*** $p = (2x)$ ***and*** $q = (5y^2)$
- $(p+q)^3 = 1p^3 + 3p^2q + 3pq^2 + 1q^3$
- $(2x - 5y^2)^3 = 1(2x)^3 + 3(2x)^2(-5y^2) + 3(2x)(-5y^2)^2 + 1(-5y^2)^3$
  $= 8x^3 - 60x^2y^2 + 150xy^4 - 125y^6$

Assignment: Holt Algebra 2, Pages 745-746 (12-56 even)

Review and Practice for Test:

Text Review Assignment: Holt Algebra 2, Pages 750-754 (2-84, even)
Text Test Assignment: Holt Algebra 2, Page 755 (1-32 all)
Practice Test: Handout, Unit 5 Practice Test, Form (C)

Test:

Unit 5 Test, Form (D)