Mathematics in Motion

STEM ACTIVITIES WITH TI-NSPIRE

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STEM ACTIVITIES WITH TI-NSPIRE

MATHEMATICS IN MOTION INTRODUCTION

Modelling things in motion – with some smart TI-Nspire mathematical tools

This book in the 'STEM (Science, Technology, Engineering and Mathematics) activities with TI-Nspire' series is, as they say, a game of two halves.

One half of the book, Section A, is a guide to some of the new exploratory tools now available for working with models and data using TI-Nspire software and/or handhelds. Instead of having to learn the concepts and techniques of differential and integral calculus before using them to model 'real-world' applications such as objects in motion, the approach is to set up dynamic examples from which basic ideas of rate-of-change, and accumulated values can be explored numerically and graphically as an introduction to (and motivation for) the calculus. It assumes that teachers and students have hands-on access to TI-Nspire technology.

The other half of the book, Section B, is a set of possibly free standing investigations into modelling motion based on different forms of data collection: manual, video and data logging. It uses examples of experiments which you perform outdoors or in a sports-hall – or model experimentally in the laboratory or classroom. In each of these, the scientific principles (mainly from physics) and the mathematical modelling techniques are closely integrated. The key link is with activities which students themselves like to engage – either putting themselves or some object into motion – and are thus intended to maximise the interest, relevance and understanding of what is going on. The booklet by itself can suggest activities and approaches, but it needs to be accompanied by resources such as photographs, video clips, newspaper cuttings, data sets, TI-Nspire documents etc. Some useful accompanying resources can be found at www.nspiringlearning.org.uk.

CONTENTS

Introduction

This booklet is aimed to introduce readers to some aspects of mathematics developed by key historical figures like Kepler (1571-1630), Galileo (1564-1652), Newton (1642-1727) and Leibniz (1646-1716) to help solve some of the important problems about how things move. Galileo is credited with a famous experiment in dropping things of different size and weight from the top of the Leaning Tower of Pisa and recording that they all took the same time to hit the ground. Newton is famous for coming up with a theory of constant gravitational attraction after an apple hit him on the head. We will investigate the progress of a ball dropping to ground from the top of the Pisa tower – a fall of just under 56m. How fast do you think it will be travelling when it hits the Earth, and how long will it take? The number of accurate scientific instruments in Galileo's time would have been rather limited!

In fact it was problems of this kind that led Newton and Kepler independently to invent a branch of mathematics called **calculus**. We will use problems like this to find out something about what calculus is and how it can be applied. We will assume that you have come across some graphs of simple functions like:

$y = 10$, $y = 5x-3$ and $y = x^2 – x$. Now mathematicians are known to have a love/hate relationship with computing technology! While they are often banned from exams, calculators and computers have been designed and used by mathematicians – for example both Pascal and Leibniz built mechanical calculators, and Alan Turing designed the Colossus computer at Bletchley Park to crack the German Enigma code during the Second World War.

How would you find the square root of 1000? Well once upon a time children used to have to learn a pencil-and-paper process to do this themselves – but most mathematicians would have used things called "tables of logarithms" and most engineers would have used a 'slide rule' (some probably still do!). I guess you've got an electronic calculator with a square root button? The first such calculator came out in the 1970s and cost nearly £100.

So we are going to explore some of the built-in functions of the TI-Nspire as our new scientific instruments. The main ones we will use are:

manipulation to change the shape and position of a linear, quadratic or sine graph;
regression to find the equation of a function which gives a good fit to some data points;
tangent to draw a line at a point on a graph which just touches the graph;
slope which measures the gradient of a line and
integral which measures the area under a graph between two points on the x-axis.

So let's get back to the dropping ball.

Section

A

TI-Nspire and
mathematical
techniques

Chapter 1

Ready, steady, drop! Modelling a falling ball.

We'll assume we have a stop-watch set to zero which will be started at the time the ball is released. Also that at the release the ball is momentarily still, and has fallen zero distance! So the only force which acts upon the ball is the Earth's attraction downwards, giving it an acceleration of **g** downwards.

What units is **g** measured in, and how big is it?

We will assume that when the ball falls there are no other forces acting on it, like wind or air resistance, and also that the value of **g** does not change appreciably in 56m.

So it feels like time to set up a spreadsheet on TI-Nspire. While the processes illustrated here all work fine on the hand-held units, many of the screenshots are from the computer software display which can show more detail.

To begin, open a Lists & Spreadsheets page. First we have entered the text 'g' in cell A1, and the value '9.81' in cell A2 (the acceleration in ms^{-2}). We will need to set up an interval by which the time ticks by – this is called 'dt' and initially holds the value 0.1 s. **fig_01**

Now we can store these values as variables. With the cursor over cell A2, select the variables button **var** and from the menu select **Store Var**, then enter 'g'. Similarly store the value in cell A4 as the variable 'dt'.

At the top of the column 'B' enter 'time', and enter '0' in cell B1. Enter the formula '=b1+dt' in cell B2 – the 'dt' should turn bold to show it is the name of a variable. With the cursor over cell B2 from the **Data** menu select **Fill Down**. Now move the cursor down over the set of cells you want to fill with this formula e.g. from B3 to B12.

We have now defined the list 'time' to hold {0, 0.1, 0.2, ..., 1.1}. Do the same for column C to store the list 'accn' – this time enter the formula '=g' in cell C1 and fill down from C2 to C12. Finally set up a list called 'zero' in column D – first enter '0' in D1 and then use it to fill down from D2 to D12.   Now we are ready to see the results graphically. **fig_02**

A		B time	C accn	D zero	E
1	g	0	9.81	0	
2	9.81	0.1	9.81	0	
3	dt	0.2	9.81	0	
4	0.1	0.3	9.81	0	
5		0.4	9.81	0	
6		0.5	9.81	0	
7		0.6	9.81	0	
8		0.7	9.81	0	
9		0.8	9.81	0	
10		0.9	9.81	0	
11		1.	9.81	0	
12		1.1	9.81	0	

fig_01
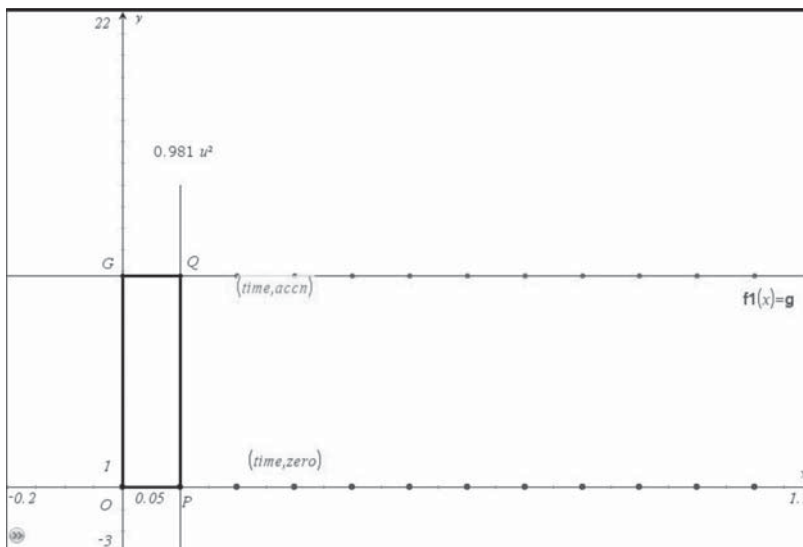
fig_02

On a new Graphs screen set the Window Settings as shown. Select the **Scatter Plot** tool from the **Graph Type** menu, and select 'time' for the x-axis and 'accn' for the y-axis of the plot s1, and also 'time' for the x-axis and 'zero' for the y-axis of plot s2.

Now select **Function** from the Graph Type menu and enter 'g' for the value of the function f1(x). Of course the graph of y=g should pass through one of the sets of data points. If you want you can select the first Scatter Plot and use **Attributes** to select a different symbol for the data. Use the **Perpendicular** tool from the **Constructions** menu to construct a line perpendicular to the x-axis through the first data point P(dt, 0). Construct the **Intersection Point** of the perpendicular with the graph y=f1(x) and call it Q. Construct the Intersection Point G of the graph with the y-axis, and the Intersection Point O of the x- and y-axes. Use the **Polygon** tool from the **Shapes** menu to click in turn on P, Q, G, O and P again. Finally select **Area** from the **Measurements** menu and measure the area of the rectangle PQGO. We know that the area is given by the base (dt) multiplied by the height (g) – and so we can check that its value is g.dt. In fact in order to compute the speed (or velocity) after 1.2 seconds, say, we just need to find the total area under the graph y=f1(x) from x=0 to x=1.2 . The speed will given by 1.2g. The value 1.2 is in seconds, and g is in metres per second per second, so the resultant is in metres per second - the correct units for speed.

Now it is time to go back to the spreadsheet. You could copy the first page of your document and paste it after the second page if you like. **fig_03**

| A	B time	C accn	D zero	E iv	F vel	G
					=cumulativ	='g*time
1 g	0	9.81	0	0	0	0.
2 9.81	0.1	9.81	0	0.981	0.981	0.981
3 dt	0.2	9.81	0	0.981	1.962	1.962
4 0.1	0.3	9.81	0	0.981	2.943	2.943
5	0.4	9.81	0	0.981	3.924	3.924
6	0.5	9.81	0	0.981	4.905	4.905
7	0.6	9.81	0	0.981	5.886	5.886
8	0.7	9.81	0	0.981	6.867	6.867
9	0.8	9.81	0	0.981	7.848	7.848
10	0.9	9.81	0	0.981	8.829	8.829
11	1.	9.81	0	0.981	9.81	9.81
12	1.1	9.81	0	0.981	10.791	10.791

fig_03

We can set up column E to hold the list 'iv': the increments in the velocity after each time step. Just enter '0' in cell E1, '=g*dt' in cell E2 and **Fill Down** from cells E2 to E12. In order to find the velocity at any time we just need to add up the appropriate number of increments – corresponding to adding up rectangles under the y = f1(x) graph. There is a useful TI-Nspire list function called 'cumsum', short for 'cumulativesum'. Enter 'vel' at the top of list F, and the formula 'cumsum(iv)' into the formula cell below it.

As a check we also know this must give the same result as just multiplying the time values by g. In column G enter the expression 'g*time' in the formula cell.

You will see that TI-Nspire isn't sure which 'g' to use in a cell – either the variable 'g' holding the value 9.81 or the column G itself – so a "Conflict Detected" dialogue box appears. Select 'Variable Reference' for g. Now copy the Graphs page from page 2 and paste it below page 3 as page 4.

Select **Scatter Plot** from the **Graph Type** menu, and select 'time' as the x-axis and 'vel' as the y-axis for the plot s3. Surprise, surprise – the points all lie on a straight line through the origin O. **fig_04**
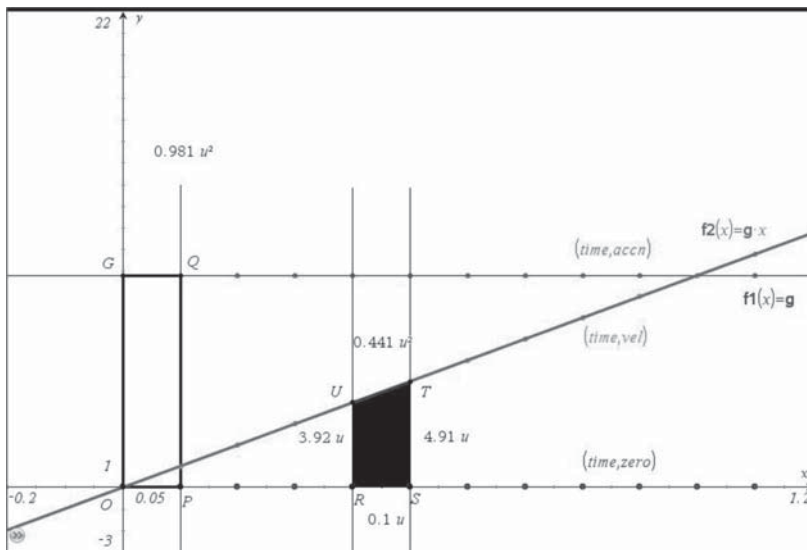
fig_04

What is its slope?
Can you suggest its equation y = ???

This is the key graph for modelling motion – the **velocity-time** graph.
The **slope** at any point on it measures the rate of change of the velocity – which is the **acceleration** – and the **area** underneath it measures the change in **distance**, as we shall see.

The graph f2(x) = g*x has been drawn. (Note that we use the x-axis to represent time and the y-axis to represent different things, like acceleration and velocity.) The points R (0.4, 0) and S (0.5, 0) have been constructed on the x-axis. Perpendiculars to the x-axis from each have been drawn to intersect the graph y = gx at points U, T, and the quadrilateral RSTU constructed. What shape is RSTU? We can measure the lengths of its sides RU, RS and ST, and also its area. We can change the Attributes of the polygon to shade it in. The area is 0.441 – what units should that be measured in?

Maybe you know that the area of a trapezium is the base multiplied by the average of its two end heights – check that this agrees with the value TI-Nspire has found. The average height is a velocity in metres per second, and the base is a time in seconds. So their product is a distance measured in metres. Areas of trapezia like RSTU represent the increases in distance in each time step dt. So again we can return to the spreadsheet to explore the distances travelled. We'll use a slightly different means of setting up the increases in distance 'id' in column H. Enter the formula '=dt*(f1+f2)/2' in cell H1 and use it to fill down from H2 to H11. (Note that TI-Nspire uses lower-case names like f2 for the contents of cells like F2.) **fig_05**

A		B time	C accn	D zero	E iv	F vel	G	H id	I dist
◆						=cumulativ	='g*time		
1	g	0	9.81	0	0	0	0.	0.04905	0
2	9.81	0.1	9.81	0	0.981	0.981	0.981	0.14715	0.04905
3	dt	0.2	9.81	0	0.981	1.962	1.962	0.24525	0.1962
4	0.1	0.3	9.81	0	0.981	2.943	2.943	0.34335	0.44145
5		0.4	9.81	0	0.981	3.924	3.924	0.44145	0.7848
6		0.5	9.81	0	0.981	4.905	4.905	0.53955	1.22625
7		0.6	9.81	0	0.981	5.886	5.886	0.63765	1.7658
8		0.7	9.81	0	0.981	6.867	6.867	0.73575	2.40345
9		0.8	9.81	0	0.981	7.848	7.848	0.83385	3.1392
10		0.9	9.81	0	0.981	8.829	8.829	0.93195	3.97305
11		1.	9.81	0	0.981	9.81	9.81	1.03005	4.905
12		1.1	9.81	0	0.981	10.791	10.791		5.93505

fig_05

The values in column H are the areas of the trapezia under the velocity-time graph and are equal to the average velocity in the interval multiplied by the time taken. Now set up column I to hold the 'dist', for the distances fallen. Enter '0' in cell I1 and the formula '=i1+h1' in cell I2. Use cell I2 to fill down from I3 to I12.

Copy the latest Graphs page and paste it below the latest spreadsheet. Set up a new Scatter Plot showing 'time' against 'dist'. **fig_06**
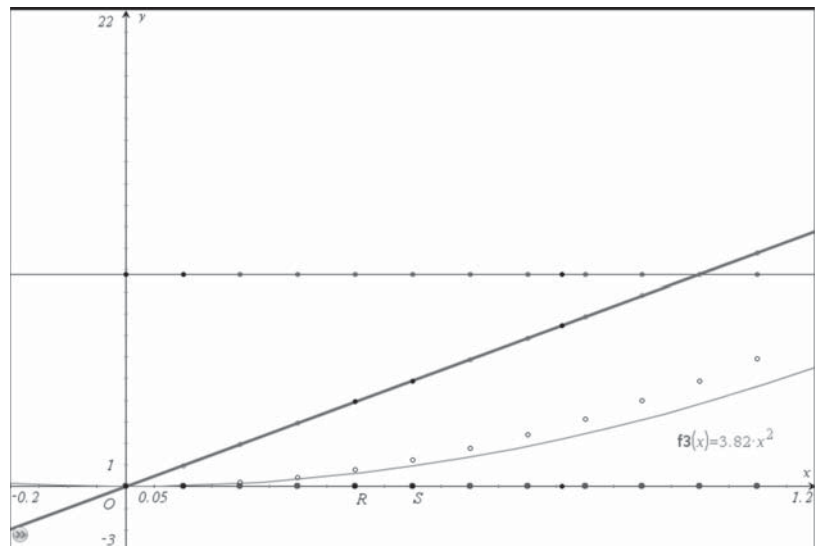
fig_06

It looks as if the points on this latest scatter plot lie on the graph of a quadratic function. Select Function Plot and enter f3(x) = x^2. Now you can **manually manipulate** this function to change the position of its vertex and/or the slopes of its 'jaws'. Try dragging a point on the graph not too close to O and see what happens to the formula for f3(x) as you flex the graph upwards towards the distance-time data points. It looks like f3(x) = $5x^2$ is a good fit (On the screen shot f3(x) = $3.82x^2$ currently.)

Another approach uses a statistical calculation, called a regression model, to find the formula of a function which may be a good fit to the data. Let's use a **Quadratic Regression** on the 'time'-'dist' data to see how well that fits. The result of doing a **Stat Calculation**, such as a quadratic regression, is that the equation of a 'best fit' function is stored in a graphing function e.g. in f4(x), which we can graph in a Graphs page, and which we hope will match well to the data points which we can show on a Scatter Plot. This technique will be frequently used when we're exploring models, not just of motion.

So, once more unto the spreadsheet, dear friends! Perhaps it's time to insert a fresh one. Set up 'time' in the A column and 'dist' in the B column. From the Statistics menu select Stat Calculations and Quadratic Regression. **fig_07**

Now select the appropriate entries for the Dialogue Box: X List is 'time', Y List is 'dist', Save RegEqn to f4, 1st Result Column c[]. You will see that cell D3 holds 4.905, D4 holds -3.E-13, cell D5 holds 1.E-13 and cell D6 holds 1. The 'scientific notation' -3.E-13 means -3×10^{-13}, which is a very small number indeed (less than a trillionth!), so both b and c are effectively zero. This leaves the quadratic function as f4(x)=$4.905x^2$. Where do you think the number 4.905 comes from? The value of 1 in cell D6 is a statistical measure of how close a fit the function is to the data, with 1 being a perfect fit!
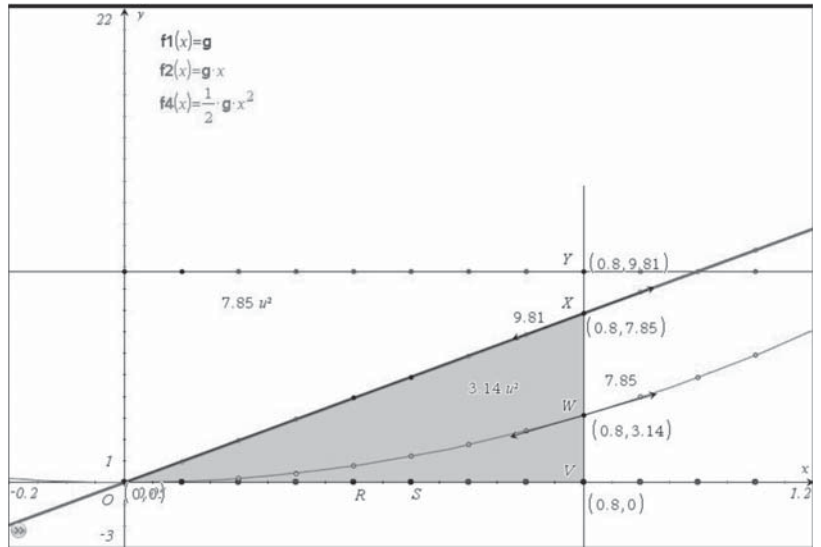
So let's go back to the (copied and pasted) Graphs page to review what we have found.

First hide unwanted detail. Enter f4(x) = ½*g*x2, and hide the graph of f3(x).

So we have the acceleration function f1(x) = g, the velocity function f2(x) = g*x and the distance function f4(x) = ½*g*x^2. At any time, like x = 0.8s, the intersections with the perpendicular to the x-axis meet the graphs in W, X and Y. The point X lies on the velocity-time graph showing a velocity of 7.85 ms^{-1}. We can construct the Tangent to f2(x) at X from the Points and Lines menu. Measuring its Slope we get a value of 9.81, which is the corresponding value of the acceleration in ms^{-2}. Measuring the Integral under f2(x) from O to V we have the shaded area as 3.14 u^2. This corresponds to the distance travelled, in metres. The u^2 is shown because it shows the area is measured in square units – we need to look at the number and appreciate that the physical quantity it represents is a distance in metres. The slope 9.81 of the segment OX gives the gradient of the velocity graph which corresponds to the acceleration (i.e. g). The integral of f1(x) from O to V gives the area 7.85 under the acceleration graph corresponding to the velocity, which is also the slope of the tangent to f4(x) at W. **fig_08**

	A time	B dist	C	D	
•				=QuadReg	
1	0	0	Title	Quadrati...	
2	0.1	0.04905	RegEqn	a*x^2+b*...	
3	0.2	0.1962	a	4.905	
4	0.3	0.44145	b	-3.E-13	
5	0.4	0.7848	c	1.E-13	
6	0.5	1.22625	R²	1.	
7	0.6	1.7658	Resid	{-1.E-13,...	

fig_07

fig_08

So now we can solve our problem about timings in Pisa using graphs rather than spreadsheets.

The problem was to find how long it takes a ball to fall 56m to the ground from the top of the Leaning Tower of Pisa, and the velocity with which it hits the ground. Use a suitable Window, show the graphs of f1(x), f2(x) and f4(x) and draw the graph of f5(x) = 56. **fig_09**
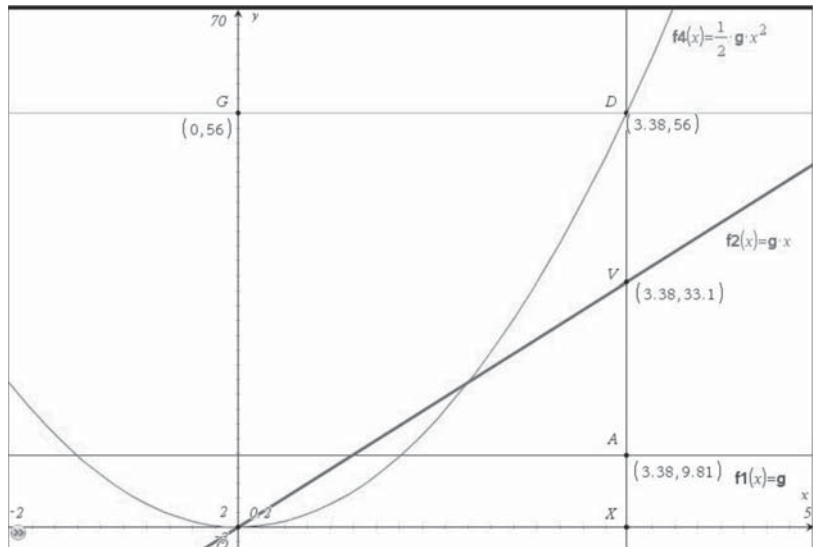
fig_09

Construct the Intersection Point D of the graphs of f4(x) and f5(x), and the Perpendicular line through it to the x-axis. Construct its Intersection Points V, A with the graphs of f2(x) and f1(x) and also the x-axis. Now can you read off the answers from the graphs? As an alternative to a graphical solution, we could use algebra, and solve the equation f4(x)=56 i.e. ½ g x^2 = 56, so x^2 = 112/g = 11.42, and x = 3.38 to 2 d.p., indicating it takes 3.38 seconds to fall 56 metres.

Substituting this value for x in f2(x) gives y = 3.38 g = 33.15 which means 33.15ms^{-1}. Remember the **acceleration** function is **constant**, the **velocity** function is **linear** and the **distance** function is **quadratic**.

So now we have met the main modelling tools available to us in TI-Nspire. Next we will look at some TI-Nspire techniques to put models into motion.

Chapter

2

Ready, steady, go.
Setting things in motion.
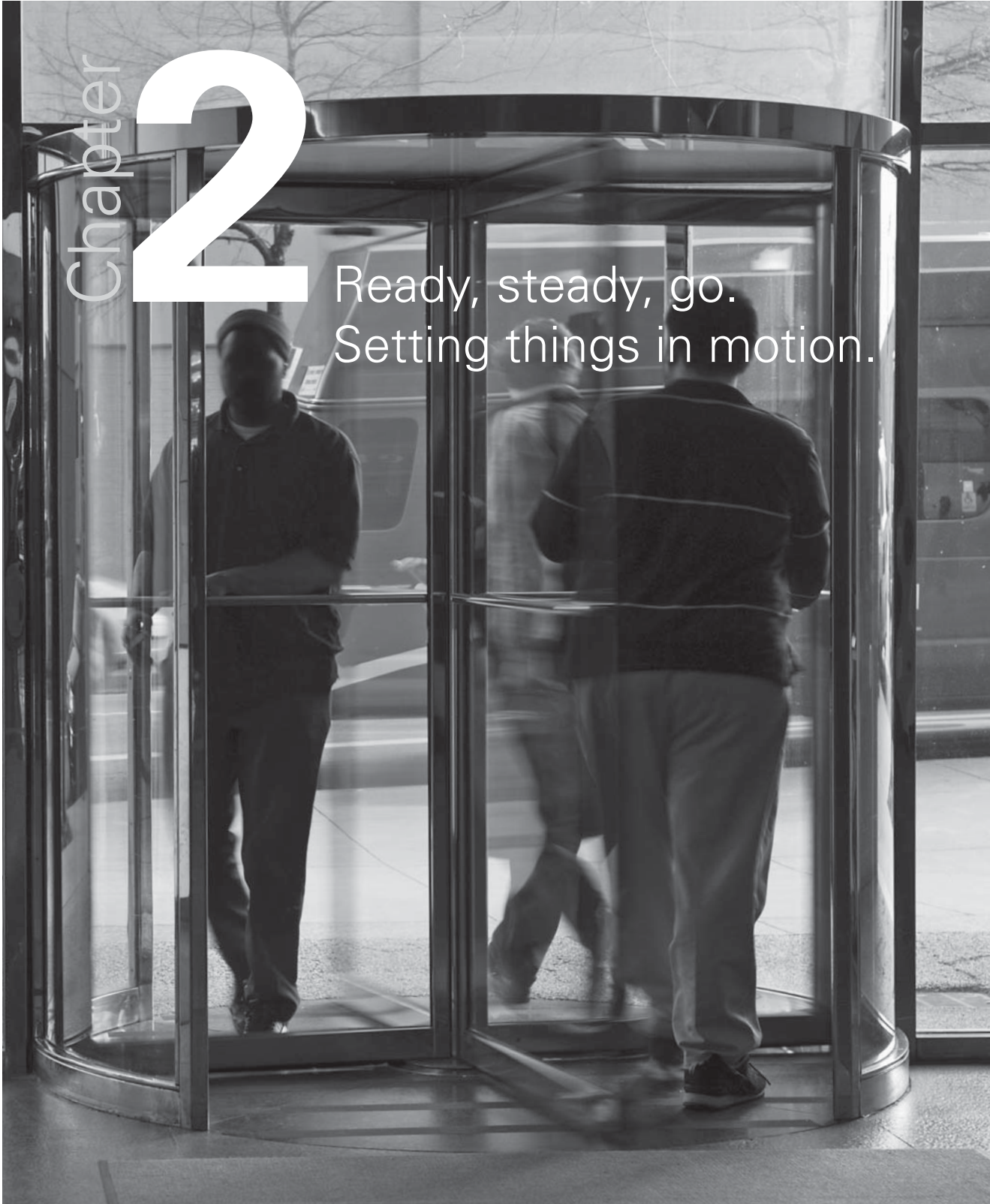
We now know the predicted values of the velocity and distance fallen at any time t – still assuming no air resistance, wind, or ball-catching birds! We can put this into action by using the Animation feature in TI-Nspire.

To start you might want to use a copy of the last page from section 1.1 and hide the coordinates, D and V and graph f1. Now construct a Segment OX between the origin O and the point on the x-axis at the end of the fall, X. Construct a Point T on it. Construct the Perpendicular to OX at T to meet the velocity graph f2 at D and the distance graph f4 at V. Construct the Vector DT. G is where Galileo is standing! So OG represents the tower. (You need a vivid imagination.) From the Transformations menu select Translation and use the vector DT to translate the point G to the point B (for ball) on the y-axis. **fig_10**
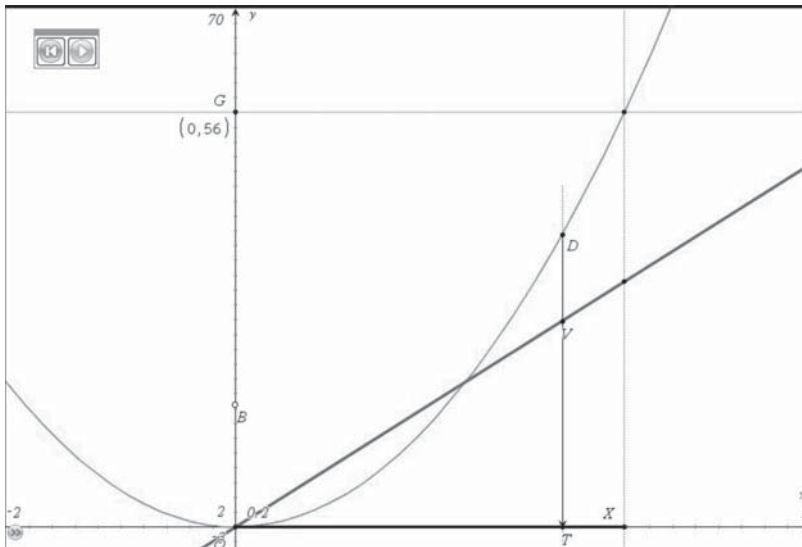
fig_10

You can change the Attributes of B to make it a bigger shape. Now check that as you slide T on AX so the point B slides on GO. Open the Attributes for point T and enter '2' for the 'unidirectional animation speed'. Press Enter to confirm, and then use the Animation Controller to set it going. If things work OK you should see point B speed up as it falls.

In fact we now have the basis of an animation for any kind of **projectile**. That is something which when moving is only acted on by the single force of gravitational attraction downwards. This should apply (ignoring spin, air resistance etc) to throwing a ball (netball, basketball, cricket,...), hitting a ball (cricket, tennis, hockey, golf,...), kicking a ball, throwing other objects, firing objects etc. Of course the object (e.g. ball) may not be at rest – in which case the increases in velocity will be added on to the initial velocity, u say, rather than the 0 we assumed for Galileo. Similarly the distances may not be measured from 0, but from an initial position, P, say. So let's explore what difference this makes to the acceleration, velocity and distance functions. Well, it won't make any difference to the acceleration, which we assume is still constant: f1(x) = g. Actually it will, since we have been rather loose about upwards and downwards, positive and negative.

Motion upwards makes distances get bigger and motion downwards makes them get smaller. If you hit a tennis ball upwards the acceleration due to gravity causes it to slow down and to stop climbing, then to start to come back to Earth. So the initial velocity was positive, but the effect of acceleration was to make the upward (positive) velocity get smaller, then become zero and then to become increasingly negative.

So now we say that the acceleration due to gravity is **g** = -9.81 ms^{-2}. With that in mind we can build a different picture on a new Graphs page. In fact it would make sense either to start a New Problem (which will stop us getting mixed up with any variables we defined but no longer use), or, perhaps better, Close the current File and start a new one. **fig_11**
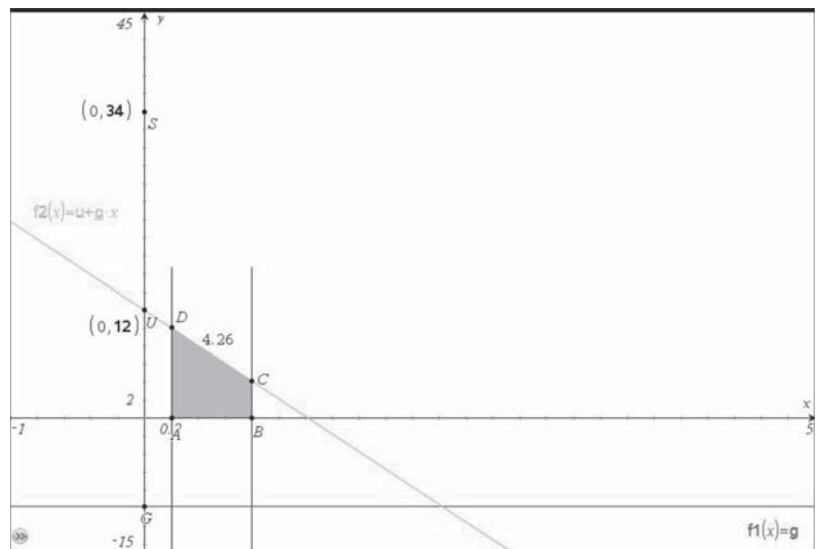
fig_11

Use Text to enter -9.81 and then Store it as the variable **g**.
Define the function f1(x)=g.

This should now be a horizontal line (constant) below the x-axis. Find its Intersection G with the y-axis. You will want to change the graph window to suit.

In order to set up an initial velocity, u, Construct a point U on the y-axis, use Coordinates & Equations to show its coordinates, and Store its y-coordinate in the variable **u**. The velocity function is now f2(x)=**u**+**g***x. This is a straight line with negative slope, since g is negative.

The increases in distance will be given by the areas between the graph of this function and the x-axis.

Construct two Points A,B on the x-axis, construct Perpendiculars to the x-axis through them, and their Points of Intersection C, D with the graph of f2(x). Either use Integral (from the Analyze menu) to find the area under f2 from A to B, or construct the polygon ABCD and measure its area. In either case we have a trapezium and we can check that the area is indeed the base multiplied by the average of the end heights, as before. From now on it will be safer to use 'Integral' rather than 'Area' for measuring areas under a graph.

This is because when an area is completely below the x-axis it needs to be measured as a negative number, but 'area' only computes positive numbers. Construct a Point S on the y-axis to set up the initial distance and store its y-coordinate as s0. **fig_12**

A	B time	C acc	D iv	E vel	F id	G dist
1 g	0	-9.81	-4.905	12	4.77375	34
2 -9.81	0.5	-9.81	-4.905	7.095	2.32125	38.7738
3 u	1.	-9.81	-4.905	2.19	-0.13125	41.095
4 12	1.5	-9.81	-4.905	-2.715	-2.58375	40.9638
5 s	2.	-9.81	-4.905	-7.62	-5.03625	38.38
6 34	2.5	-9.81	-4.905	-12.525	-7.48875	33.3438
7 dt	3.	-9.81	-4.905	-17.43	-9.94125	25.855
8 0.5	3.5	-9.81	-4.905	-22.335	-12.3938	15.9138
9	4.	-9.81		-27.24		3.52

fig_12

So now we can rebuild a spreadsheet with our new data. Enter '=g' in cell A2, '=u' in A4, '=s0' in A6 and a value for dt, e.g. '0.5' in A8. Store the value in this cell as the variable dt. The cell D1 holds '=g*dt' and this is copied down to D8. (Equally we could have entered 'g*time' in the formula box above D1.) Enter '=u' in E1, '=d1+e1' in E2 and copy this down to E9. So now we have the velocities – which are indeed decreasing. To find the increases in distance in column F enter '=(e1+e2)/2*dt' in F1 and copy down to F8. Enter '=s0' in G1, '=f1+g1' in G2 and copy down to G9. **fig_13**
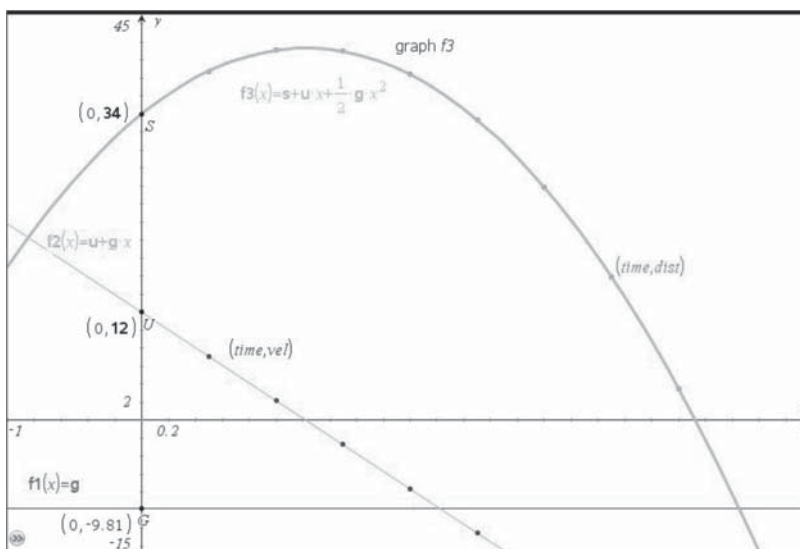
fig_13

Returning to the Graphs screen we can show the scatter plots of 'time' against 'vel' and also 'time' against 'dist'. The velocities agree with our linear function f2, and the distances look like a quadratic function but passing through S (0,s0). So enter f3(x)=s0+u*x+1/2*g*x2 as the distance equation and check it passes over the scatter plot.

The important thing to remember about all our graphs so far is that the x-axis tells us about 'time' and so the graph f3(x) is the distance-time graph etc. Now these functions can model someone hitting or throwing a ball upwards.

The next trick is to combine two forms of motion! If you throw, hit or kick a ball at angle upwards it will have two kinds of motion in one! Horizontally, gravity has no effect, and the only force would be a resistive force of friction – which we will assume to be negligible (for now, at least) – so the motion will just be that of constant velocity horizontally. Vertically we assume it has the form of motion we just investigated. If we now change variables so that x represents horizontal distance, y represents vertical distance and t represents time, then we can use y = y0 + uy*t + ½*g*t2 for the vertical distance function, where y0 is the initial vertical height and uy is the initial vertical velocity. Similarly we can use x = x0 + ux*t for the horizontal distance function. The mathematical trick which allows us to combine both these functions is called a **parametric equation**. We have to specify a set of values for the parameter t, such as 0≤t≤5.
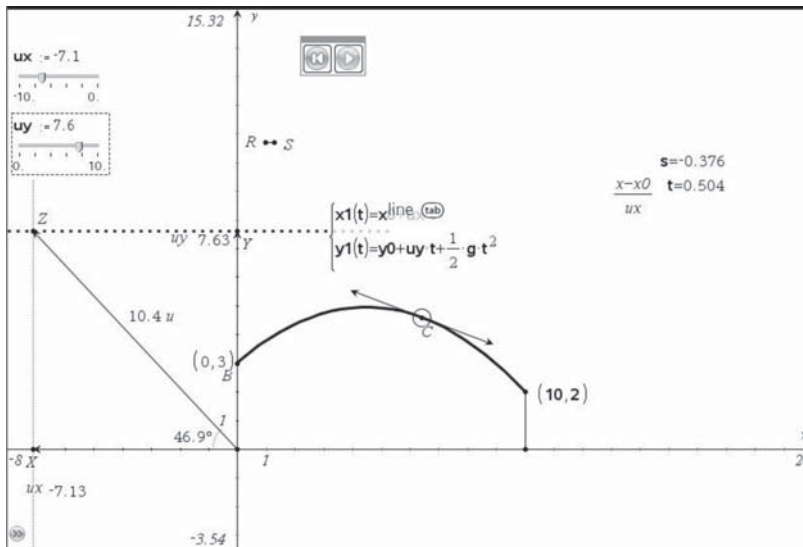
In order to bring a bit more realism to the task we will now try to model a throw at netball or basketball. **fig_14**

On a new Graphs page, Zoom Square, then create a Point to represent the thrower and measure its coordinates. Store these as x0 and y0. You can construct the Perpendicular from the point to the x-axis, find the Intersection, construct the Segment and hide the line – to show a 'stick' person. The y-coordinate represents the height 2m of the release of the ball from the hand – so it can be higher than the thrower. Insert two sliders to set the values of the initial velocities ux and uy. Adjust the Properties of ux so that it is negative – i.e. so that the ball is thrown towards the y-axis (right to left). Construct a Point B on the y-axis and drag it to represent the height of the net/basket. Insert the formulae for the Parametric function x1(t),y1(t) – using values for t from 0 to 1 in steps of 0.1. **fig_15**

fig_14
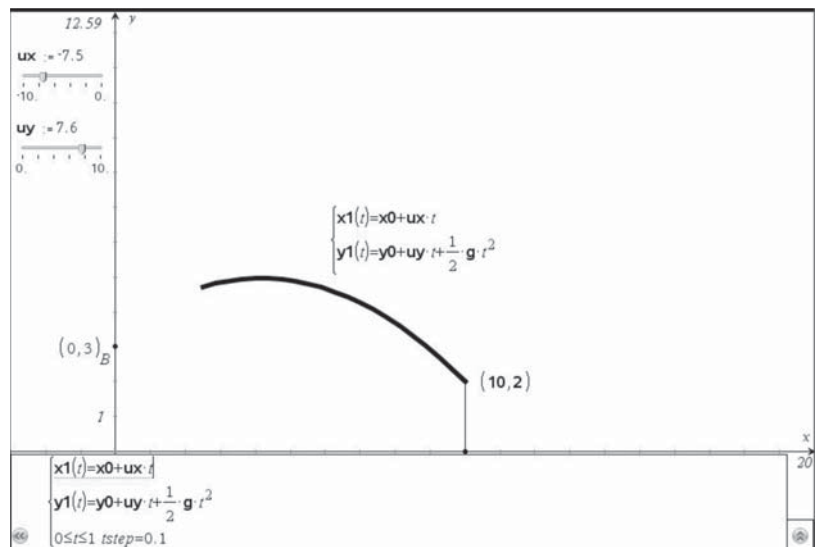
fig_15

With luck you should get a graph which starts from the thrower's hand and looks as if it might land quite close to the target. Edit the upper value for t, e.g. to 1.5 to see more of the flight-path (also called the **trajectory**). **fig_16**

15.32

ux := -7.1

-10.   0.

uy := 7.6

0.   10.

R ↔ S

s=-0.376

x−x0   t=0.504

ux

$$\begin{cases} x1(t)=x \text{line} \\ y1(t)=y0+uy\cdot t+\dfrac{1}{2}\cdot g\cdot t^2 \end{cases}$$

Z

uy 7.63 Y

10.4 u

(0,3)
B

(10,2)

46.9°

1

-8 X

ux -7.13

1

20

-3.54

fig_16

Now we can slide to change either ux or uy or both to see what might make a good throw. Construct a Point C on the graph and draw the Tangent at P. Measure the slope and Store it as s. Find the Coordinates of C and store them as x and y. In order the find the value of t we are going to have to solve an equation. Use Text to enter the expression '(x-x0)/ux' and then select the Calculate tool. Highlight the text expression and each time it asks for a value to use, press 'L' to use the stored values. Press Enter to show the result and Store it to the variable t. Check you can slide C along the curve. In order to make a more realistic ball Construct a Segment RS. Use the Compass Construction tool with the Point C and the Segment RS to draw a circle centre C. Adjust the length of RS to modify the radius. If the ball doesn't look circular use Zoom Square. Finally you can change the Attributes of Point C to give it an animation speed. Can you calculate the maximum height reached and at what time? Collect some x- and y-coordinates from points on the parametric graph and see if you can fit a Quadratic Regression to them of the form $y = ax^2+bx+c$. Can you find any relationships between the coefficients a, b, c and our given variables like g, x0, y0, ux, uy?

The initial velocities seem rather large, don't they? We can use the Text tool to enter the expressions 'ux' and 'uy' and the Calculate tool, with L to retrieve the values of the variables ux and uy. Use the Measurement Transfer Construction tool to make a Point X on the x-axis at (ux,0) and a Point Y on the y-axis at (0,uy). Construct Vectors OX and OY. Construct Perpendiculars to the axes at X and Y to meet at Z, and then construct the Vector OZ. The Length of OZ is the launch speed of the ball, and the Angle XOZ is the launch angle.

Can you adapt the ideas to make more accurate simulations of balls (and other objects) in motion? At what sort of speed does a basket ball player launch a free throw? How about a lob at tennis? How fast is a long-jumper's launch, or a shot-putter's throw? Can you redesign the approach so that it is the launch speed and angle which you vary, and from which you calculate ux and uy?

# Chapter 3

Areas under quadratics – a curve with a kink.
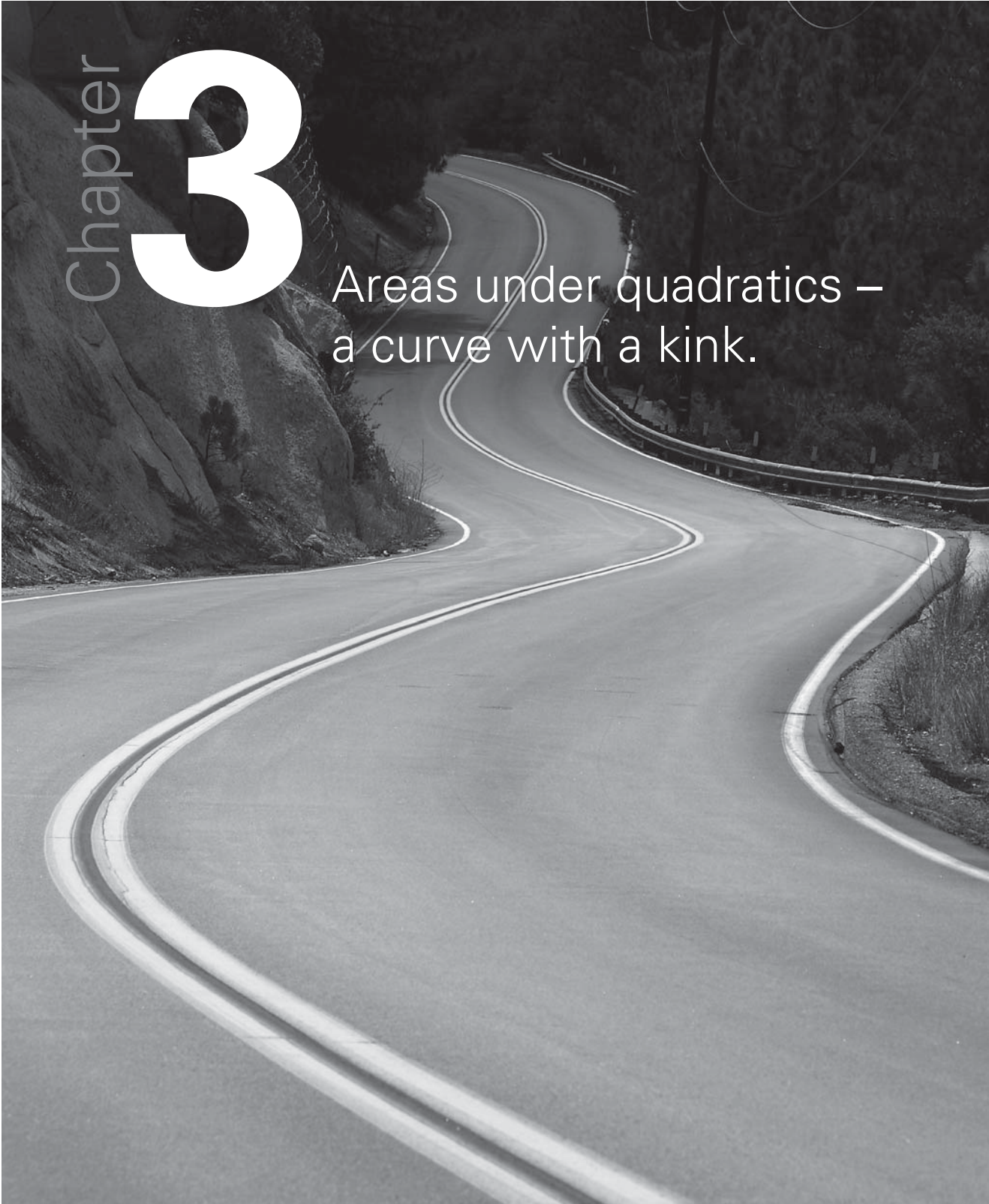
Before we go further, it will useful to add another kind of function and its graph to our armoury to go along with constant, linear and quadratic functions. We'll start with a quadratic function of the form $y = ax^2+bx+c$ and investigate areas between a few points. Actually Archimedes did something similar well over 2,000 years ago (**http://en.wikipedia.org/wiki/The_Quadrature_of_ the_Parabola**) as did Johannes Kepler (1571-1630), Bonaventura Cavalieri (1598-1647), James Gregory (1638-1675) and Thomas Simpson (1710-1761)! All of which suggests that quadratic curves are pretty important tools in our understanding of physical science.

We'll make a fresh start with a Spreadsheet. In column A enter labels and values for a, b and c and Store them. In column B enter a few x-values. In column C apply the quadratic function $y = ax^2+bx+c$ to the 'xd' list. Set up column D with zeros. **fig_17**

Now bring up a new Graphs page. Draw Scatter Plots of xd against yd and of xd against zero. Enter the formula $f1(x) = ax^2+bx+c$ and measure the Integrals under f1 between each pair of data points on the x-axis. Store the areas as variables **a1**, **a2** and **a3**. **fig_18**

A		B xd	C yd	D zero	E
			='a*xd^2+'b*xd+'c		
1	a	0		3	0
2		2	1	0	0
3	b	2	1		0
4		-5	3	6	0
5	c				
6		3			

fig_17
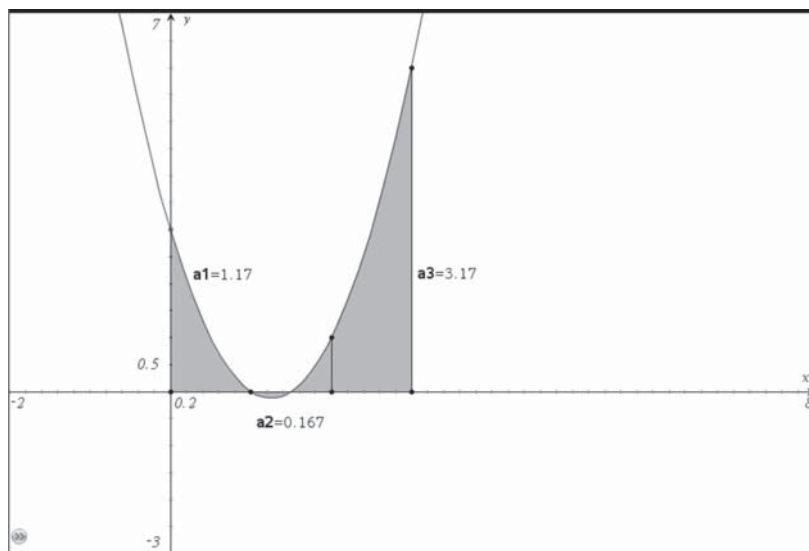
fig_18

Enter these as the values of the cells in column E which will be the increases in areas. Enter an initial value of 0 in cell F1, then '=f1 + e1' in F2, and copy down to E4. So we now have just four points on the graph of the integral of the quadratic. Following the pattern we will assume the result is a cubic, and then seek to confirm it. So carry out a Cubic Regression on 'xd' and 'zd' – which will be stored in f2(x). Do any of the values of the coefficients of the cubic in column H look familiar? Can you convert H3 and H4 into fractions? Do either have any connections with cells A2, A4 or A6? **fig_19**

	A	B xd	C yd	D zero	E area	F zd	G	H
◆			='a*xd^2+					=CubicReg
1	a	0	3	0	1.16667	0	Title	Cubic Re...
2	2	1	0	0	0.166667	1.16667	RegEqn	a*x^3+b*...
3	b	2	1	0	3.16667	1.33333	a	0.666667
4	-5	3	6	0		4.5	b	-2.5
5	c						c	3.
6	3						d	0.
7							R²	1.
8							Resid	{0.,2.4E-...

fig_19

Well the x^3 coefficient of the regression cubic is 2/3 which looks like 1/3 **a**, the x^2 coefficient is -5/2 which looks like ½ **b** and the x coefficient is 3 which looks like **c**.

So let's check out the cubic f3(x) = 1/3***a***x^3 + ½***b***x^2 + **c***x for size. Of course it passes through the four data points.

The quadratic function, f1(x), Intersects the x-axis in the points A and B. The perpendicular bisector of AB cuts the x-axis in C, cuts the graph y=f1(x) in D and cuts the graph y = f3(x) at G. Perpendiculars to the x-axis through A and B cut y=f3(x) in E and F. Tangents are constructed at D, E, F, G and their Slopes measured. What can you say about each of these points? **fig_20**
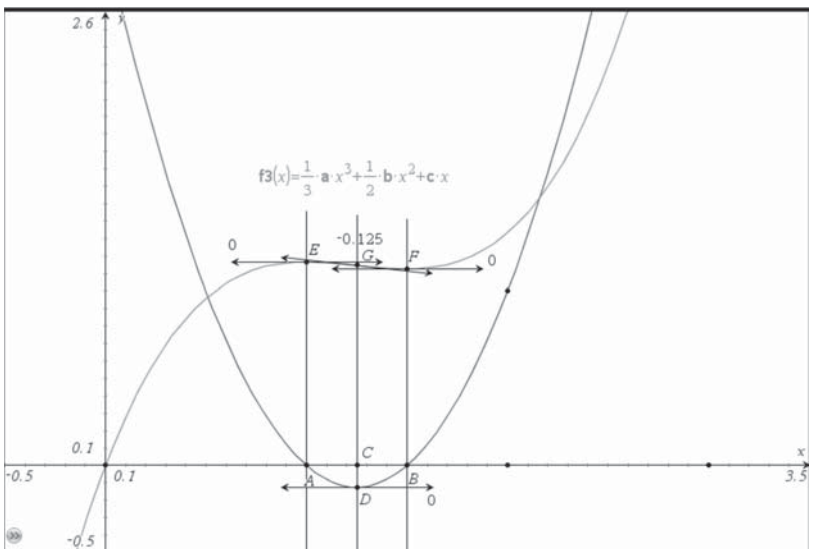
fig_20

At A and B the values of f1(x) are both zero. Since y = f1(x) is the graph of the slopes of y = f3(x) this means that E and F must have zero gradient, and hence correspond to a **maximum** or **minimum** (usually!). Now think of the traversing the graph of y=f3(x) from left to right like a climbing roller coaster! From O to E the climb is getting less steep until at E we start to descend for a bit until the steepest descent at G. Then we switch to continue to descend

less steeply until F where we start to climb again. This special point G is called a **point of inflection** and corresponds to a slope of zero on y = f1(x) i.e. the **vertex** of the quadratic – which is where the acceleration would be zero.

Experiment with different values of a, b and c to check that we have indeed got the correct function in f3(x). Calculate some more values for the lists xd, yd, zeros and zd. Check that the cubic regression still gives a value of 1 in H7 (or very nearly). Will every cubic function have (a) a maximum, (b) a minimum, (c) an inflection?

That little excursion into graphs, tangents, slopes, areas and integrals was intended to give a taste for the branch of mathematics which Newton and Leibniz independently created, known as **calculus**. If you are taking AS, A2 or Higher mathematics you have probably met some calculus already – in which case we hope maybe that these explorations help make more sense of it. If you haven't yet met any calculus we hope it might whet your appetite to find out some more. However people like Archimedes, Kepler and Cavalieri worked out ways of finding integrals (area functions) for quadratic curves using geometry – the basis of a technique now called Simpson's Rule.

You can see this for yourself by inserting a New Problem and opening a Graphs page. Insert Sliders to control the values of the variables **a**, **b** and **c**. Adjust their Settings so that they can be negative, zero or positive. Graph the Function f1(x)=**a***x^2+**b***x+**c**. Construct two Points A, B on the x-axis together with Perpendiculars to the axis through them. Construct their Midpoint C and Perpendicular Bisector of AB. Construct the Intersection Points D, E, F of the three vertical lines with the graph y=f1(x). Construct a Perpendicular line through F, the middle of these, to the y-axis and find its Intersection Points G, H with the other two verticals. **fig_21**
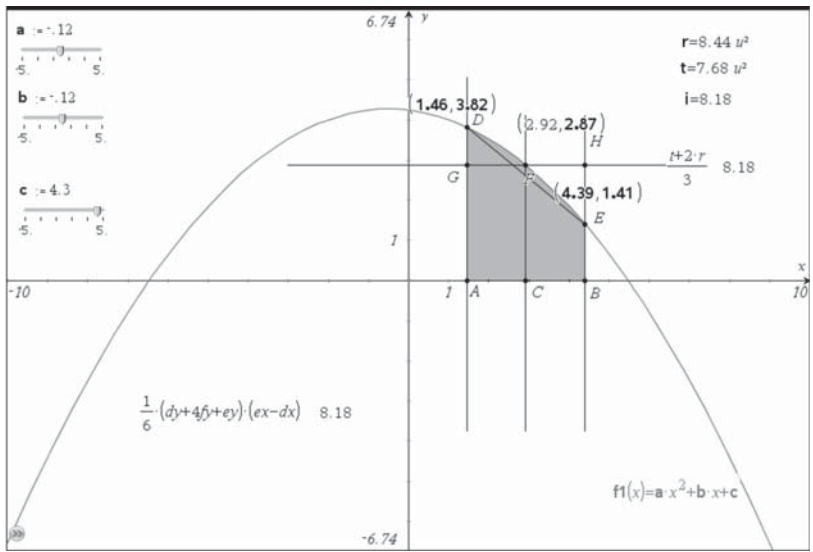
fig_21

Construct the Polygons ABHG and ABED (what shapes are they?), measure their Areas and store them as **r** and **t**. Measure the Integral of f1 from A to B and store it as **i**. The discovery was that the area under the quadratic graph from A to B is 2/3 of ABHG plus 1/3 of ABED – i.e. the weighted average of the rectangle and trapezium in the ratio 2:1. Use the Text tool to enter the formula '(t+2*r)/3' and the Calculate tool to evaluate this using the stored variables (just press L each time you are prompted). Store the coordinates of D in **dx** and **dy**, and similarly for E and F. Can you find the area **i** in terms of these 6 coordinates? It's that formula which is known as Simpson's rule.

Now it's time to put our skills and tools to the test in modelling various kinds of (mainly) human movement including something far less well explored and documented – that of human athletic performance in the 100m sprint.

Section B

Investigating motion from data

Chapter

4

Constant velocity on skates.

Newton's First Law of motion states that when a body is in motion it will continue to move at a constant speed in the same direction unless acted on by a force. So, can we find potential examples of constant velocity? These could include walking, running, cycling, sailing, swimming etc. at a constant speed – but in each case there is a propulsive force overcoming some form of frictional force. An extreme example is someone descending on a parachute, or of a snowflake, where the vertical force of gravitational attraction is exactly balanced by the air-resistance – and so the object has reached its 'terminal velocity'.

So we might look for areas where frictional forces are more or less negligible, such as in a vacuum – or more practically on ice or snow, or maybe on roller blades? YouTube has a BBC science video 'P039: Forces' which shows some examples where the presenter roller-skates down a ramp and appears then to travel with steady speed for a few metres: **http://www.youtube.com/watch?v=RQwYXYhUsgg**.

This is something you could easily set up and test for yourselves. You could mark lines at 1m intervals and use stop watches to measure times taken to reach 1m, 2m, 3m etc. The data recorded could then be entered directly into a spreadsheet, such as a Lists & Spreadsheet page of a TI-Nspire document. You could also set up a data logger, such as one or two CBR2s, to record the distances travelled at a set time interval e.g. 1/20 s.

In the following example we have downloaded the video from YouTube to RealPlayer in a Flash video format (.swf) and converted it to an Apple Quicktime movie format (.mov). Then we can play it back in the free Tracker 3.1 video analysis Java applet (or other video capture software) to track the position of the roller-skater (**http://www.cabrillo.edu/~dbrown/tracker/**). Tracker was written for the US Open Source Physics Project by Doug Brown of Cabrillo College and can be used either locally or web-based.

The video clip is recorded at 25 frames per second and the full clip has nearly 2500 frames. The first step is to load your video and then to use the Clip Controller (the rightmost icon under the screen) to set the start and finish frame numbers for the analysis, and the number of frames to advance between each data capture. The values chosen were to use every fifth frame between frames #877 and #977, i.e. 20 frames in all at steps 0.2 s apart. This corresponds to a section showing her from overhead travelling a distance of about 8m horizontally from the foot of the ramp. **fig_22**

fig_22

The next step is to calibrate the scene using the Ruler icon shown on the toolbar above the video area as a blue double-arrowed line. Drag the icon to where you want the measure to be. Drag out the ends and adjust the direction (change the colour if necessary). Replace the existing measurement with your own measurement e.g. 8.0 m.

The final bit of preparation is to insert axes (shown in purple on the toolbar) so that the origin is in the desired place and the direction of the axes accords with your own choice. Now you collect the data.
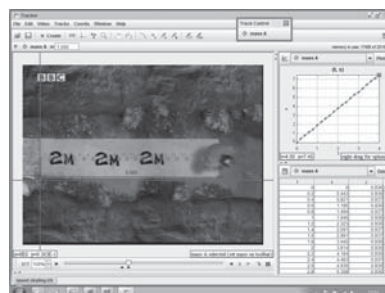
Select New Track, and Point Mass. You can collect data manually by pressing shift-click at a point in a frame. The video then advances to the next selected frame, inserts the current (t, x, y) data into a table and plots the current data in the t- and x-lists. You can select other variables to be collected and other variables to be graphed. You can also collect data automatically (auto-tracking) by selecting an easily identifiable part of the moving scene, such as the skater's helmet, and the software will then scan for nearest match in the next selected frames.   We have also selected an offset so that the tracked points start in line with the foot of the ramp.

Once data has been recorded successfully it can be copied and pasted into other software for analysis. At the moment the numerical data format used in Tracker cannot be read directly by TI-Nspire, so first paste the data into an Excel or Open Office spreadsheet. The data has been pasted into columns A, B and also D, E. The latter data has been reformatted to contain numeric data to 3 d.p.

The data can be plotted in Excel and a 'trendline' fitted – which suggests that a constant speed model looks reasonable, although at a little slower rate than suggested in the original video. **fig_23** The data in columns D and E can now be copied and pasted into TI-Nspire for further display and analysis.

The data has been entered as lists t and d in the spreadsheet. We can define column C to hold the list of results obtained by dividing each value of the distance covered, d metres, by the corresponding time, t seconds. A dialog box will open in which we will declare t and d to be the names of variables, rather than columns.

The cell D1 has been set to the mean of the cells C2 to C21. So the numeric evidence from a slightly wobbly kind of video capture is more or less a constant value for speed = distance divided by time. **fig_24**
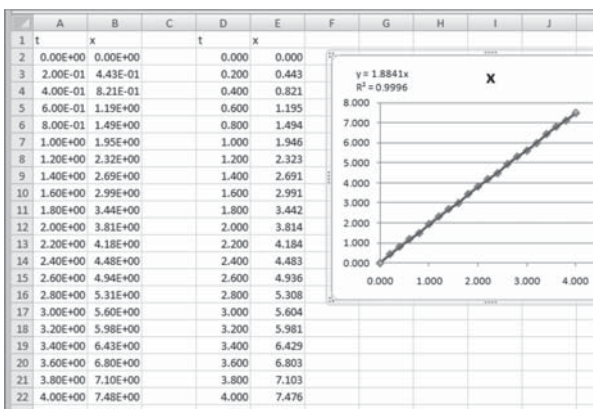
fig_23
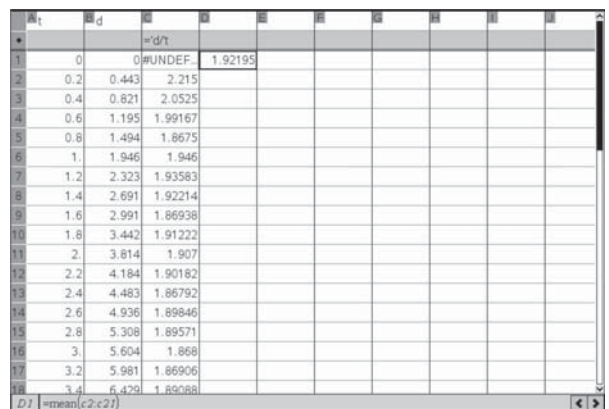
fig_24

The data can be displayed as a scatter plot in a Data & Statistics page and a Regression line plotted over it. As we can see, the slope is very similar to that computed in Excel, but the graph does not pass through the origin. **fig_25**

As an alternative, we can draw the scatter plot in a Graphs page and then plot the linear function f1(x) = x over it. Now we can use manual manipulation to change the slope of the graph without moving it away from the origin. So a good fit, by eye, appears to be f1(x) = 1.89 x. **fig_26**
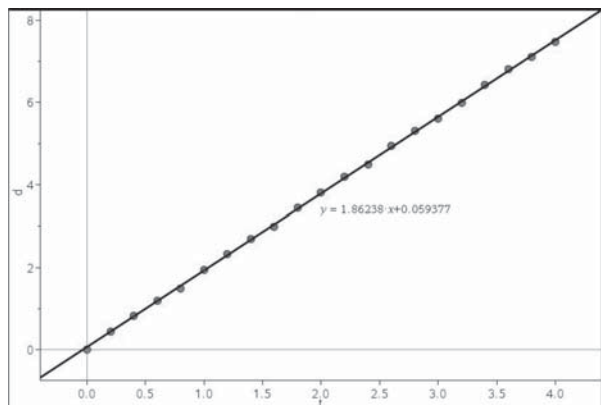
fig_25
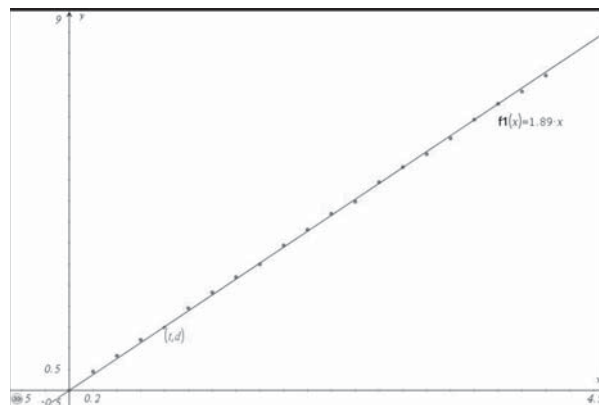
fig_26

But while the program is suggesting that the skater has a constant velocity we know that actually there is a frictional force at work which will be slowing her down, however slightly. If we try a Quadratic Regression on the data we see that the coefficient, a, of x^2 is small but not negligible. **fig_27**

Over the 4 seconds observed, the slowing up is not very noticeable, but we can see that our quadratic model suggests she would come to rest after about 2 minutes (actually 116 seconds) during which time she would have covered about 110 m. **fig_28** Can you set up your own roller skating experiment, with video capture and/or manual timing with stopwatches to explore just how far you can glide on a roller blade?

A t	B d	C	D	E	F	G
		='d/t			=QuadReg	
1	0	0	#UNDEF...	1.92195	Title	Quadrati...
2	0.2	0.443	2.215		RegEqn	a*x^2+b*...
3	0.4	0.821	2.0525		a	-0.008181
4	0.6	1.195	1.99167		b	1.89511
5	0.8	1.494	1.8675		c	0.038651
6	1.	1.946	1.946		R²	0.999769
7	1.2	2.323	1.93583		Resid	{-0.0386...

fig_27
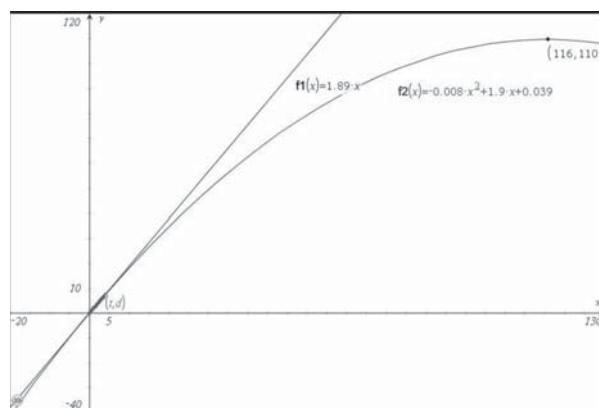
fig_28

Chapter

5

Juggling some balls
in the air.
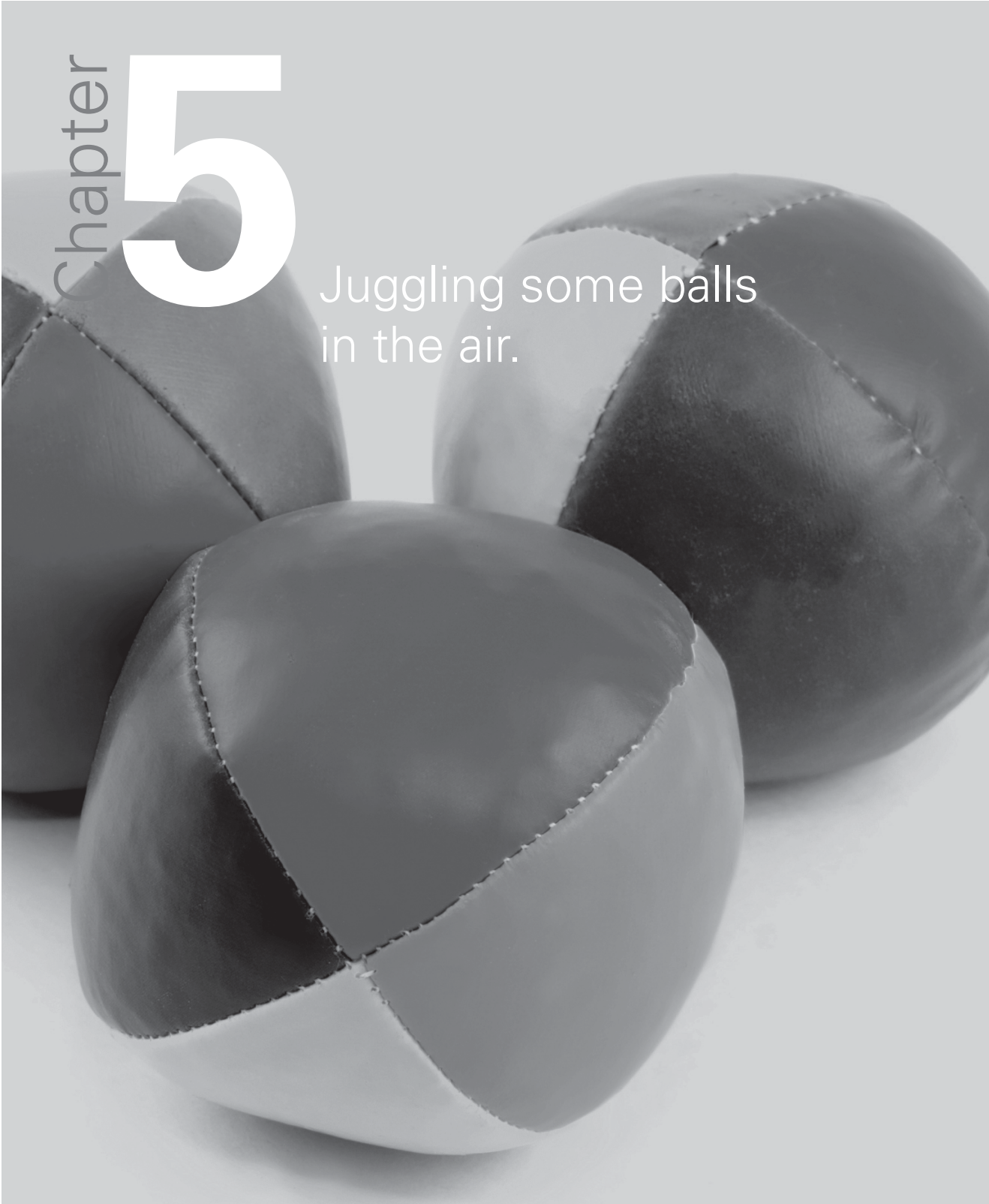
In Section A, we worked at describing motion just from a theoretical model. We assumed that a ball dropped from a height would just be subject to a single constant force, that of gravitational attraction downwards. Here we use another video clip, this time of someone juggling, to capture and analyse data about displacement, velocity and acceleration.

You can download some sample activities and resources from the 'Physics with Video Analysis' section of the Vernier website at **http://www.vernier.com/cmat/ pva.html**. These include a Quicktime video clip **jugglerclip3.mov** of a young man juggling three balls vertically. You can use Vernier's Logger Pro or Tracker 3.0 or other video software to capture data for the middle ball, say. **fig_29**

fig_29

The data for the height of the bottom of the middle ball from the point of release have been copied and pasted into TI-Nspire. The page is split between a Lists & Spreadsheets region and a Data & Statistics region. The scatter plot of vertical displacement against time certainly looks very quadratic and the Regression Quadratic gives a very good fit. We can compare the x^2 coefficient with that obtained in section 1, where in the absence of air resistance its value was $-\frac{1}{2}$ g. So the evidence from this particular clip suggests that the value of g is about -10.1 ms^{-2}, compared with the usual one of -9.8 ms^{-2}. One explanation for this could be that the video was taken down a deep mine where the value of g was greater than that at sea level, but a more likely one is that the margin of error (about 3%) is accounted for by errors in the data-capture process, such as parallax, misreading, wobbling etc.! **fig_30**

	A t	B y
1	0	0
2	0.033	0.104
3	0.066	0.193
4	0.099	0.28
5	0.132	0.343
6	0.165	0.402
7	0.198	0.448
8	0.231	0.487
9	0.264	0.513
10	0.297	0.529
11	0.33	0.532
12	0.363	0.525
13	0.396	0.507
14	0.429	0.477
15	0.462	0.431
16		
17		
18		

$E1$ ="Quadratic ?"

$$y = -5.06408 \cdot x^2 + 3.28019 \cdot x + 0.000176$$
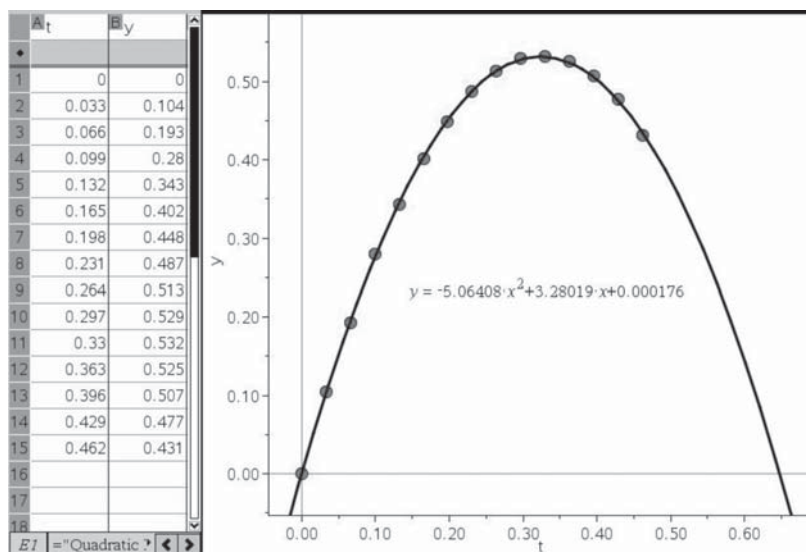
fig_30

If we use a Graphs page instead, we can draw the scatter plot and a function like $f1(x) = -5*(x-0.3)^2+0.5$, where the values have been read approximately from the previous screen. Now you can manually manipulate the graph to make a good fit to the data. Using the Geometry menus we can find the intersection points L and R of the graph with the x-axis. We can also find the point of intersection V of the graph with the perpendicular bisector of LR. We want to try to maintain L to be at the origin while adjusting the quadratic curve (a parabola) to fit the data well, as shown. **fig_31**
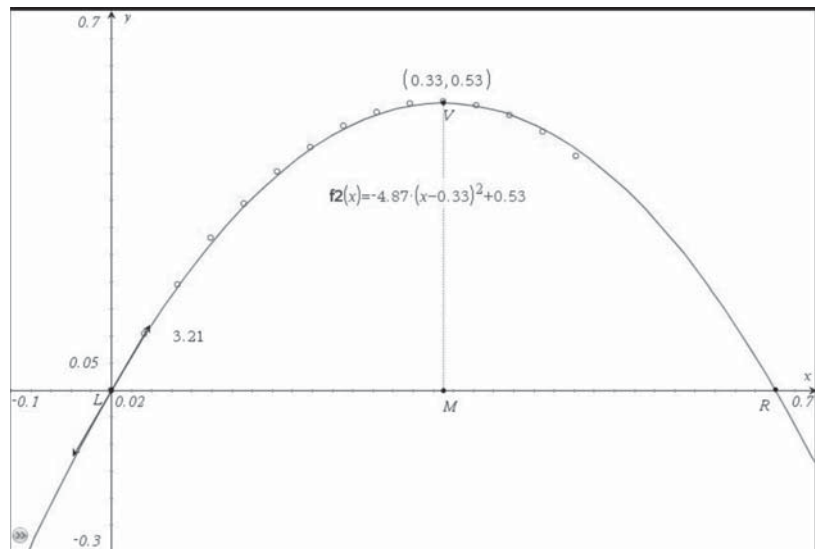
fig_31

Now we get a better approximation for g as -9.74 ms^{-2}, but probably more through luck than judgement! We can read off that the maximum height of 0.53m was achieved after a time of 0.33s from release. We can also draw the tangent to the curve at L and measure its slope to estimate the initial velocity of the ball as 3.21 ms^{-1}.

Reversing the theory of section 1, we can say that if the displacement is given by the quadratic function $f1(x) = a(x-b)^2+c$, then the velocity will be given by the linear function $f2(x) = 2a(x-b)$ and the acceleration by the constant function $f3(x) = 2a$. Draw the graphs of these functions. At what time is the velocity zero? Interpret this in terms of the video clip. When is the velocity negative? What does this mean in terms of the ball? At what speed will it land in the juggler's hand (at the point R)?

Investigate dropping, throwing and catching different types of balls. Can you collect and analyse your own data? Can you find any types of balls for which air-resistance cannot be neglected?

It should be becoming clear that mathematical modelling (i.e. the fitting of functions and graphs to data) is not an exact science, and that we must be very careful about making hard and fast statements such as "the velocity/ acceleration is constant". We must also exercise care in interpreting graphs. We have stuck with the general convention in mathematics about using x- and y-graphs, but in our examples the variable plotted on the x-axis is time (usually in seconds), while we have plotted displacements (in metres), velocities (in metres per second) and accelerations (in metres per second per second) at different times on the y-axis.

When we have motion in two dimensions, such as throwing and catching a ball, then we have to take care to distinguish between a displacement-time graph and the actual path in space (the trajectory) of the ball.

Chapter

6

Putting the ball in
the net – finding
the trajectory.
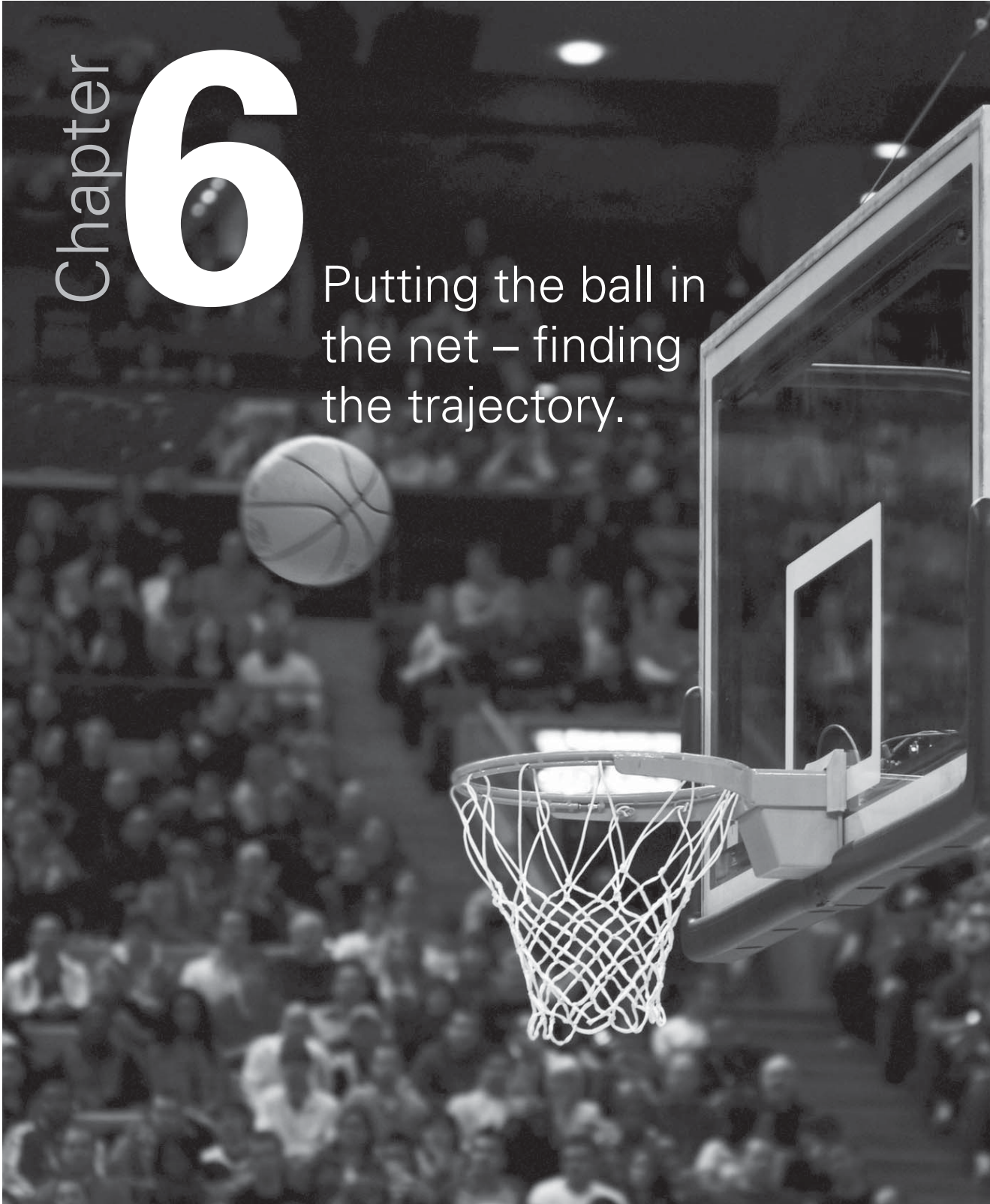
As usual, the data have been copied and pasted into a TI-Nspire List & Spreadsheets page after video data capture in Tracker **fig_32**. Scatter plots, quadratic regression and a moveable line have been used in a Data and Statistics page to show how each of the coordinates varies with the time t. **fig_33**

As expected, in the horizontal direction we appear to have constant velocity and in the vertical direction we have constant acceleration. **fig_34** In a Lists & Spreadsheets page we can compute the linear (x-t) and quadratic (y-t) regressions and store them in f1(x) and f2(x). Then in a Data & Statistics page we can plot all the variables on the same page by selecting t as the independent variable and after selecting x as the dependent variable, right click on the box to add a second variable. **fig_35**

fig_32

fig_33

The equation shown on the graph:

$$y = -5.09265 \cdot x^2 + 5.10877 \cdot x + 1.81398$$

fig_34

fig_35

In a Graphs page we can draw the scatter plots and superimpose the regression curves. Remember that in this view, the x-axis is used for time. **fig_36**

$f2(x)=-5.09 \cdot x^2+5.11 \cdot x+1.81$

(t,y)

$f1(x)=-1.67 \cdot x+-0.076$

(t,x)
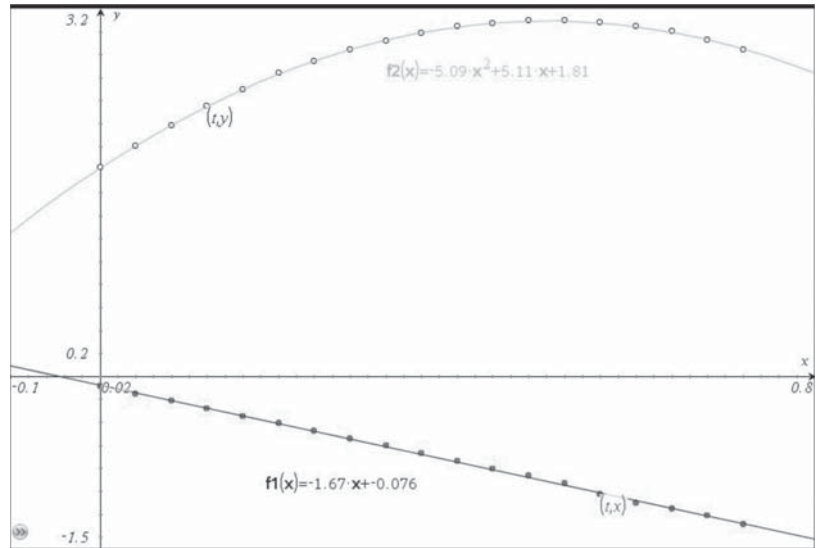
fig_36

In a new Graphs page we can draw the scatter plot of y against x – which gives us a glimpse of the geometrical trajectory **fig_37**. But now it is imperative we have equal units on each axis, so once the image is in a reasonably central position we must use Zoom Square. Now we can plot the trajectory as a Parametric curve using the linear regression function f1(t) for the x-coordinate and the quadratic regression function f2(t) for the y-coordinate. The values of t range from t = 0 when the ball is released to t = 0.75 just after the ball has entered the net. Can you now calculate the angle and velocity with which the ball was released? What value does this analysis suggest for the acceleration due to gravity g?

$(-1.19, 2.96)$

$\begin{cases} x1(t)=f1(t) \\ y1(t)=f2(t) \end{cases}$
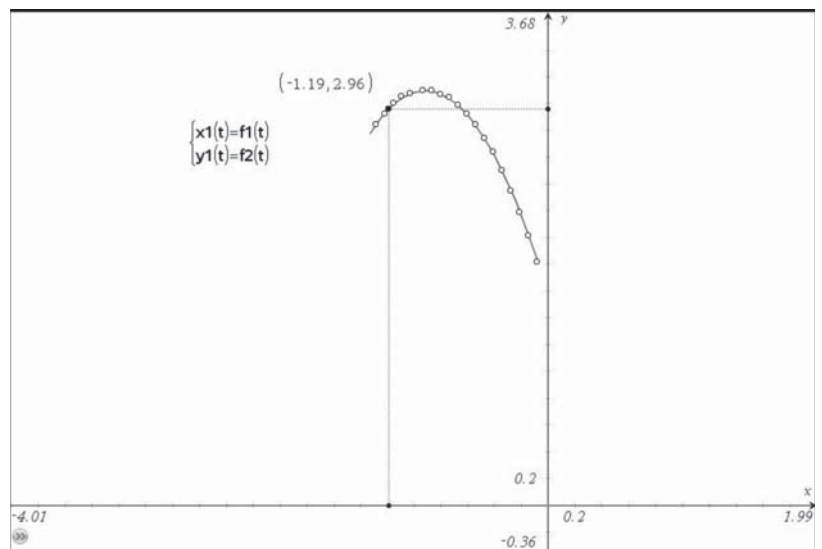
fig_37

Chapter

7

Still pictures run deep –
what can a photograph
tell us?

Birmingham Town Hall has two remarkable sculptures below it in the main square. The upper one is called The River, but is usually known as "the Floozie in the Jacuzzi"! **fig_38**

The lower one, by the same sculptor, Dhruva Mistry, is known as Youth and the figures at either side are more or less life size. Many items of mathematical software can now accept photographs as the background for graphing and drawing. These include Excel, Autograph, Cabri Geometry, Geometer's Sketchpad and the Mathematical Toolkit (Skoool). You can take your own digital photographs, find them on the Internet using e.g. Google Image search, or purchase them on CD from sources such as **http://www. problempictures.co.uk/**.

A gallery of mathematical images was sent free on CD to all English maintained secondary schools in 2005 by the DfES together with a book "Integrating ICT into the Mathematics Classroom" from the ATM. This is still available from the ATM at: **http://www.atm.org.uk/shop/products/rea025.html**. We have attached the image from the photo gallery to a line segment in Cabri II Plus **fig_39**. What mathematics can we do on a still image of objects in motion such as water spouts?

fig_38
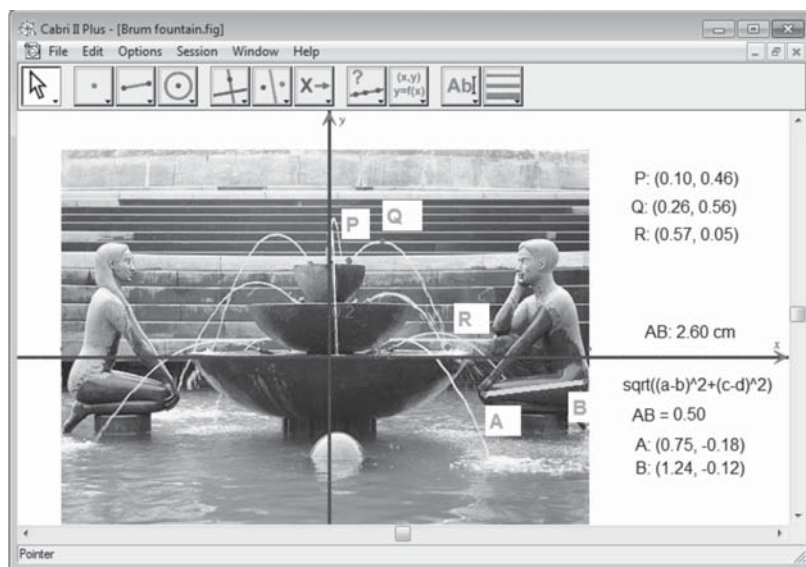
fig_39

First we collect a minimal amount of data from the photograph by rescaling axes. The distance from the knee to the buttock of the male adolescent on the right should be around 0.5m. So we set up axes and adjust scales so that coordinate distance AB is 0.5.

Then with this set of axes we have placed points at three critical positions on one of the water spouts. P is at the source, Q is at the vertex and R is at the point of entry to the basin. With a bit of ingenuity we should be able to use this data in TI-Nspire to find out more than the obvious from the picture.

This time we can just type in the coordinate data for our 3 points in a Lists & Spreadsheet page. A quadratic function ax^2+bx+c has just has three coefficients, a, b, c, so with 3 points we would have to solve 3 simultaneous equations in these 3 unknowns to find them. So if, instead we get TI-Nspire to compute a Quadratic Regression function it must be the unique function which passes through each of the 3 points. Of course this technique works for any polynomial – 2 points define a line, 3 points define a quadratic, 4 points fix a cubic etc. When we draw the scatter plot and the quadratic function on a Graphs page we can see that the Q is not exactly at the vertex. For comparison we have entered $f2(x) = 0.56 – 2*(x-0.26)^2$ which we know will have its vertex at Q (0.26, 0.56) and used manual manipulation to adjust the slopes – and as we see it just misses P and R by a small amount. **fig_40 / 41**

fig_40
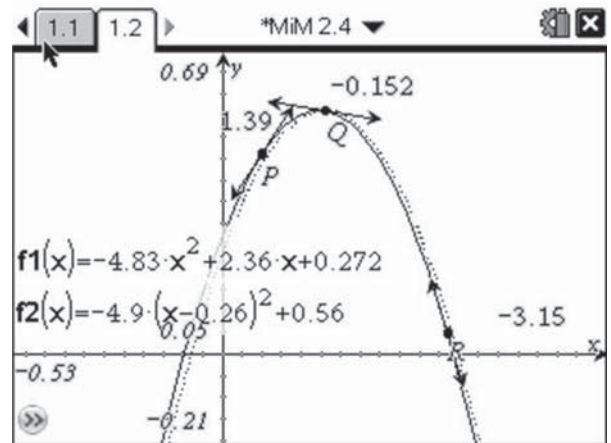
fig_41

So what have gained from this exercise? Surprise, surprise – we have found, yet again, a reasonable approximation to g – as twice the x^2 coefficient or g = -9.7 ms^{-2}. We can also draw tangents to f1(x) at P, Q and R and measure the slopes. So, for example, we now know that water leaves the spout at P at a speed of 1.39 ms^{-1}, and lands in the basin at R at a speed of -3.15 ms^{-1} – where positive speeds are upwards and negative speeds are downwards.

The point about this example is that we are looking at a continuous stream of water droplets all assumed to be moving under the same forces, and so we have same effect as we would have done if we had taken three photographs of the netball throw in quick succession (keeping the camera still, of course). That is basically what we have with a video clip – a set of still images taken at very short intervals apart.
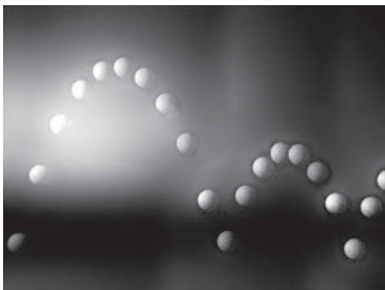
One of Richard Phillips's images on the ATM CD is a stroboscopic picture of a bouncing ball (in 2 dimensions) **fig_42**. Can you find some way to take measurements from this photograph to insert data into a TI-Nspire Lists & Spreadsheet page?

fig_42

We have no reference measurements, but the ball just might be a standard table tennis ball? If not, and we assume the value of g is -9.8 ms^{-2}, do you think you could estimate the size of the ball from this data? What do you think is the time interval between the camera shots? What percentage of the energy is lost at each bounce? How fast was the ball travelling when it hit the ground?

So for things moving very fast, like rockets say, we need a means of getting a short video clip with a very short time gap between frames. Most current forms of digital cameras, webcams and mobile phones will record video clips at frame rates between 15 and 30 frames per second (fps). Those of you who watch sport on television will now be used to seeing slow motion clips in extreme detail, e.g. of a cricket bat hitting a ball. Such cameras have been extremely expensive until recently, but now there are cameras on the market for under £300 which will take good quality videos at 210 fps, slightly poorer ones at 420 fps and even provide low resolution output at 1,000 fps. (For example the Casio Exilim range has such cameras at between £200 and £300.) This means we now collect video data from quite fast moving objects.

Chapter 8

How fast is a fast bowler?

Can you spot the ball in each of the frames above? In **fig_43** it is just leaving the bowler's hand and in **fig_44** it is just about to leave the field of view. If we know that the frames were taken about 1/20 s apart, and we guess that the ball has travelled around 2 stumps' lengths (1.5m) then we have a way to estimate the speed of the ball as about 30ms^{-1} or around 100 kph. But can we do better?

fig_43

fig_44

fig_45

The video clip: 'fast ball.avi' was recorded at 210 frames per second using a Casio Exilim EX-FH20 high speed camera with telephoto lens and imported into the Tracker 2 software. The 'tape measure' tool was used to mark the height of the stumps as 0.711m. Axes were added through the base of the stumps, and tilted through an angle of 2.2° to the horizontal. The path of the ball was tracked as a 'Point mass' from the point of release by the bowler **fig_45**. The data has been copied and pasted into a TI-Nspire Lists & Spreadsheet page for analysis.
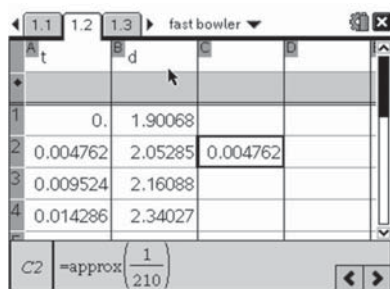
fig_46
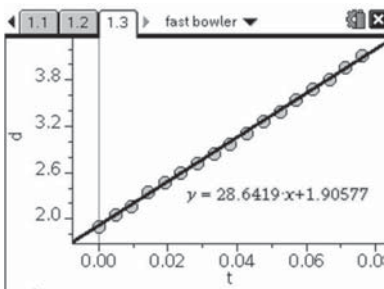
fig_47

In cell C2 we have calculated the approximate value of 1/210, which confirms the time steps shown in the t list. **fig_46**

A scatter plot in a Data & Statistics view shows the data to be well-fitted by a line, and the linear regression equation shows that the ball is travelling with a more or less constant velocity of about 28.6 ms^{-1} from an initial point of release some 1.9m in front of the stumps **fig_47**. The so-called 'popping crease' is 1.2m in front of the stumps and the bowler aims to have his back foot just behind this crease. So the other 0.7m must be accounted for by his bowling action where he leans forward and his arm is ahead of his body. The speed of about 28.6 ms^{-1} or just over 100 kmh^{-1} compares well with the range of 136-150 kmh^{-1} suggested for adult fast bowlers in: **http://en.wikipedia.org/wiki/Fast_bowling.**

Chapter

9

How fast is Juan del Potro's tennis serve?

One of us had a good seat ringside at the O2 arena in London in December 2009 when Juan Martin Del Potro played Fernando Verdasco during the Barclays ATP Men's World Tour Finals **fig_48**. They had my compact Casio Exilim EX-FC100 camera with them, but no tripod – and no flash allowed. Using the High Speed video capture at 210 frames per second they were able to shoot and edit an AVI clip at 480 x 320 pixels. With auto-tracking in Tracker they were able to get a good set of data values for one of del Potro's many first serves. Can we use the data to see how the speed of serve compares with the speed generated by a fast bowler?
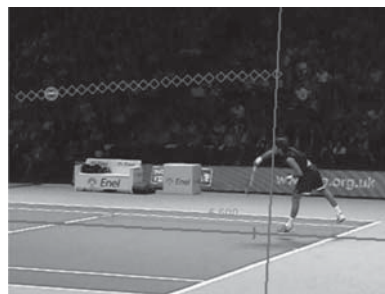
The distance from the base line to the service line is marked as 5.5m. The axes are placed just below the point where del Potro first hits the ball. The x and y data lists have been adjusted so that the ball leaves the racket at (0, 0) at time t = 0s. A new column of data, d, has been computed from $d = \sqrt{(x^2 + y^2)}$ to measure the distance travelled by the ball **fig_49**. The resulting scatter plot of d against t again shows a very close fit to a linear function **fig_50**.

Here are just four values extracted from the time t and displacement d lists. **fig_51**

Using these can you find a good approximation to Juan del Potro's service speed in ms^{-1}?

Can you convert the result from ms^{-1} into mh^{-1} (as used by the speed gun at the O2)?

Think about some other athletic, sporting and leisure activities where an object is launched. Could you use the techniques we've been working with to analyse them? Some possibilities include: javelin, shot, discus, hammer, archery, golf, hockey, football, rugby, high jump, long jump, triple jump, badminton, squash, basketball, volleyball, table tennis, Frisbee, paper aircraft, Stomp rocket, fireworks…

fig_48

fig_49
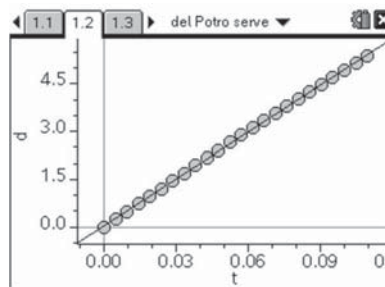
fig_50

0.	0
0.0333	1.7
0.0667	3.36
0.1	4.95

fig_51

Chapter

10

Sophie in the snow – forces on a toboggan.
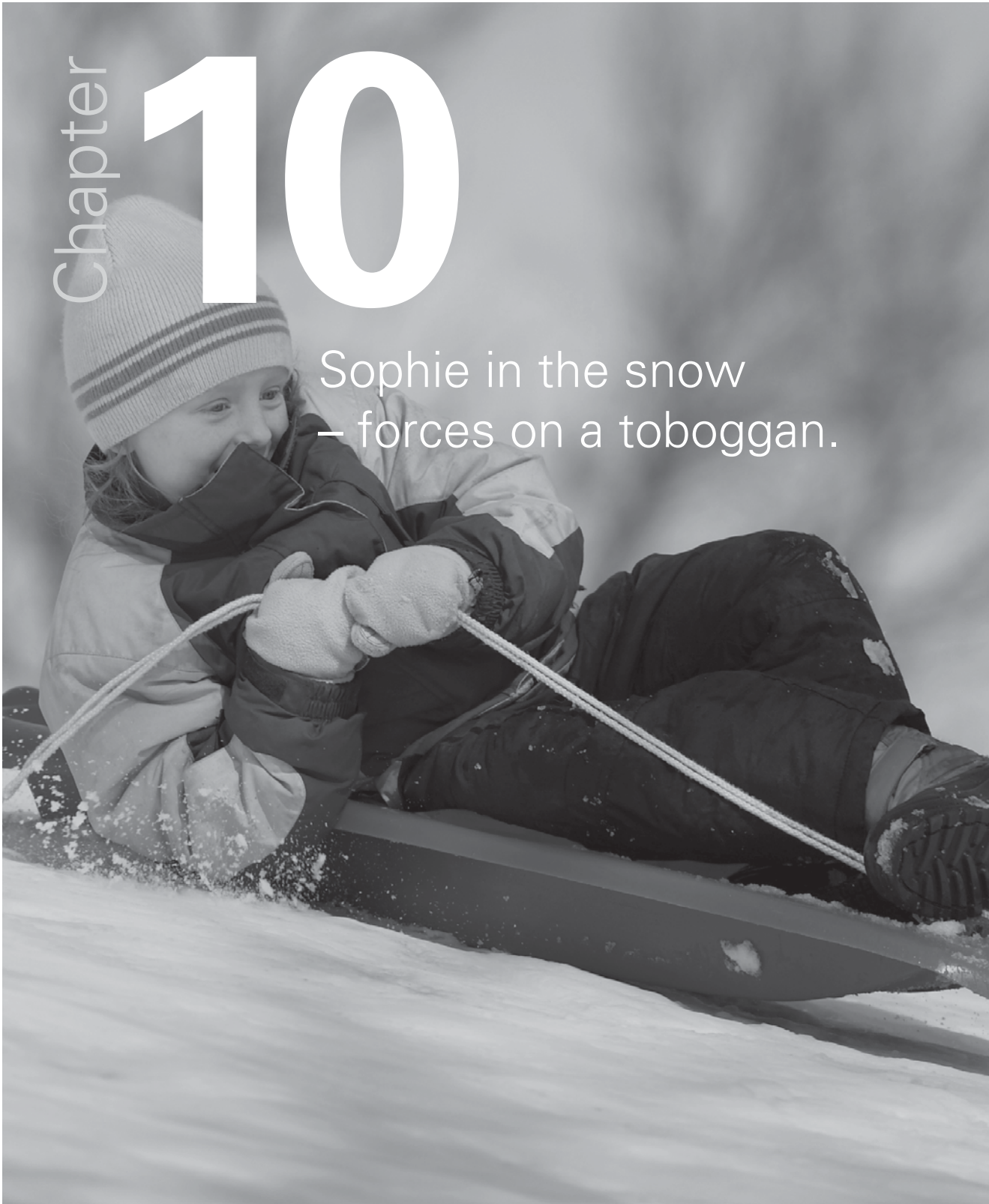
Thanks to the inclement weather in December 2009 many students were able to take to the slopes with all manner of toboggans. Our colleague Pip took a video clip of her daughter Sophie coming downhill on her 0.91m red sled **fig_52**.


fig_52

This was taken on a Casio Exilim digital camera at 30 fps and analysed using the Tracker 3 software. Using the Clip Settings button the video clip has been set to run from frame 0 to frame 80 taking data from every third frame at 30 fps. The auto-tracking has successfully picked up Sophie's foot and the front of the toboggan as target. The Tracker variable r measures the displacement of the object from the origin, so we can export the (t, x, y, r) data for further analysis. We can copy and paste the data to a Lists & Spreadsheets page in TI-Nspire **fig_53**. If we draw the scatter plot of xm and ym we need to take care over the scales. This graph is a geographical one where xm and ym are measurements of position of the toboggan in metres, so we need to use the same scales on each axis if the positions are to resemble the downward sloping hillside **fig_54**. We can see that there is quite a good linear fit, but not as accurately as in our previous models of constant velocity or acceleration. The hill isn't a man-made structure and it isn't surprising that it's not perfectly flat.

fig_53
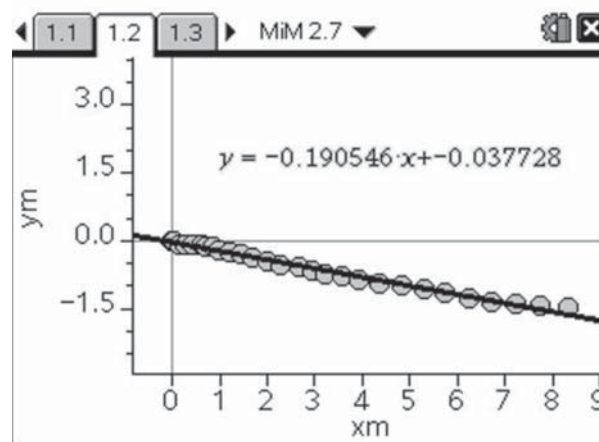
fig_54

The regression line has a slope of -0.19 which corresponds to a downhill angle of 11°. We can also look to see how well a quadratic fit approximates the displacement-time graph. Once again we have the choice of taking a Quadratic Regression, or manually manipulating the graph of a quadratic function. **fig_55**

If f1(x) gives our quadratic fit for the displacements, then we can easily find the corresponding linear fit f2(x) for the velocity using the technique of section 1. Similarly we can find the constant fit f3(x) for the acceleration. On a Graphs page we can plot all three functions. The point T is set up as a slider on the horizontal (time) axis. Its intersection with the three graphs are D, V and A. The tangent to the displacement-time graph f1(x) has been drawn at D and its slope measured – which we can see corresponds with the velocity at V. The area under the velocity-time graph has been measured as the integral under f2(x) from O to T and we can see that this corresponds with the Displacement at D. It was this fundamental link between slopes and areas which both Newton and Leibnitz developed into what we now call "calculus".
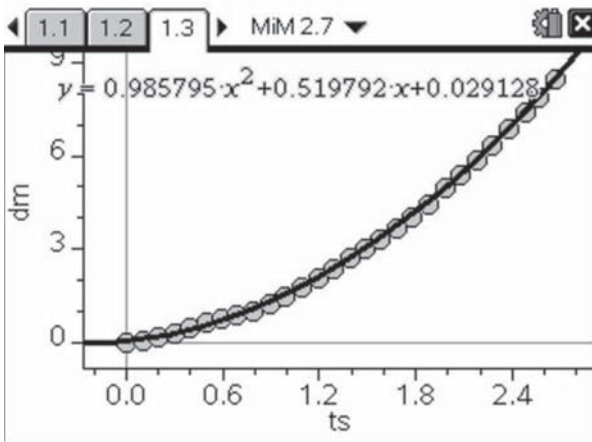
fig_55
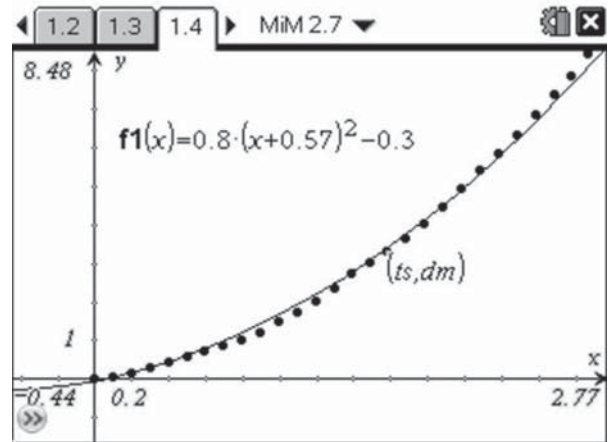
fig_56
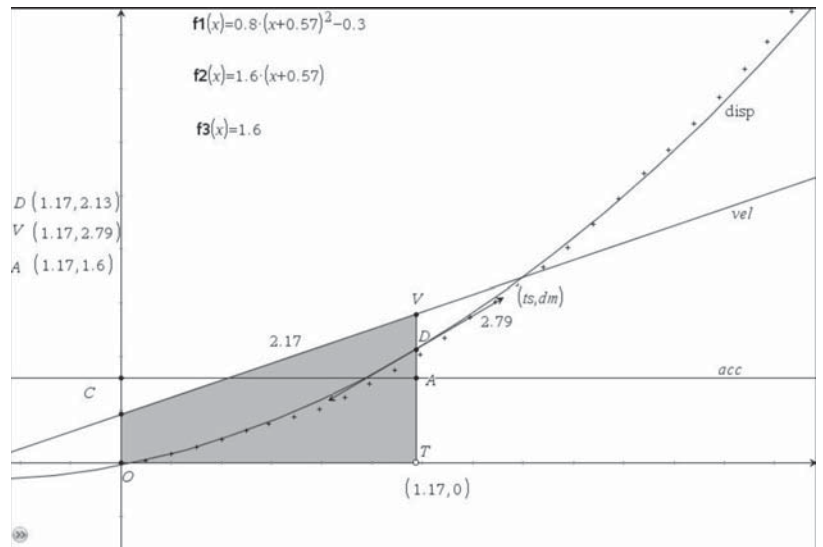
fig_57

We computed the slope of the hill as being about a constant angle $\alpha = 11°$ downwards and we can now use it to find an estimate for the coefficient of friction λ between the toboggan's runners and the icy snow.

The force down the hill is proportional to g cos α and this is opposed by a frictional force in the opposite direction proportional to λ g sin α. The equation of motion of the toboggan, ignoring air-resistance, bumps and the like is:

a = g (cos α - λ sin α)

With our experimental value of a as 1.6 we find that we get a value of $\lambda = 0.028$ for the coefficient of friction between the toboggan and the icy dry snow. And, yes, we have used a Graphs page from TI-Nspire to draw the force diagram for the toboggan.

In this approach we are, as usual, making many assumptions, like that the slope is straight, that air-resistance/wind is negligible etc. There is only one real source of possible data error – the calibration of the video. Can you find out the effect on calculating λ of, say, a 10% error either way in picking up the 0.91m length of the toboggan in the video? How can we best compensate for perspective distortion of lengths between the centre and the edges of the video frame – or are such effects negligible?

The main point here is that we are not in the realm of absolutes and right answers. This is modelling with real data and as such is how mathematics is regularly applied to complex problems. Video capture provides a cheap, portable, reliable and visual means of collecting data in the field. TI-Nspire can both help us perform complex calculations supporting mathematical modelling, but also provide means of displays and interactions which enable us to understand better the close relationship between physical quantities such as displacement, velocity and acceleration.

Now see if you can think of other examples of motion where friction is a critical force, and see if you can use video capture and TI-Nspire analysis to investigate it.
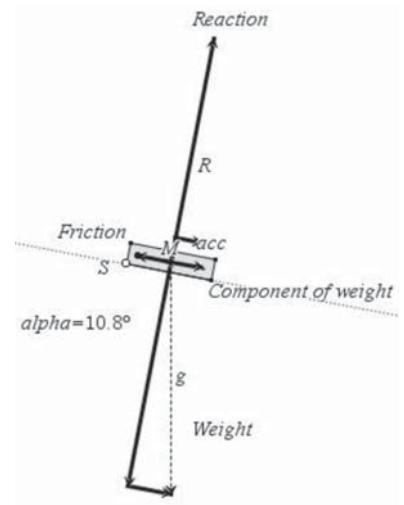

fig_58

Chapter

11

Our flexible friends – Snoopy and Ellie go bungee jumping.

Snoopy and Ellie are wooden toys condemned to spend their lives bouncing up and down suspended on the end of springs. Health and Safety stopped us using real kids for this part of the book! **fig_59**


fig_59

We made a video of Snoopy bouncing up and down over the stairs alongside a metre rule, and, as usual, ran the clip through Tracker to digitise data from a set of bounces. Now we have a different kind of force model where the downward force of the weight is increasingly opposed by the stretched spring until Snoopy comes to rest and then starts to accelerate upwards. As the tension in the spring decreases so the upwards force starts to decrease until Snoopy again comes to rest, and then starts to accelerate downwards.

The motion then repeats, more or less. So this gives an approximation to what is known as Simple Harmonic Motion. We shall see what the data has to tell us now. Here the video clip was taken at 30 fps and we see that we have nice smooth wavy ripples – looking like a sort of sine wave. In the menu of possible regression types to use there is one called "sinusoidal". **fig_60**
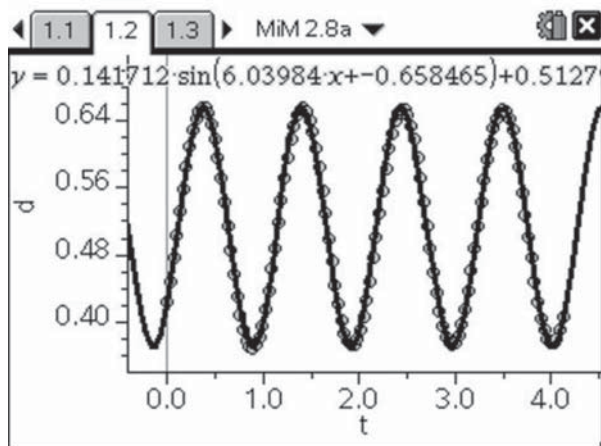

fig_60

Here we see that we get more or less 4 completed bounces up and down in a space of about 4 seconds. So Snoopy is bouncing up and down at the rate of about 1 cycle per second – which is known as a **frequency** of 1 Hertz, written 1 Hz. The time between successive bounces is called the **period** and it is about 1 s. **fig_61**
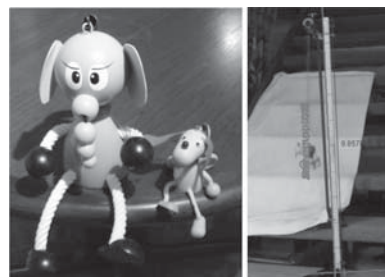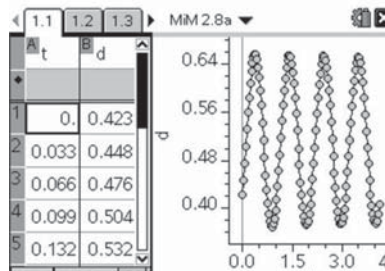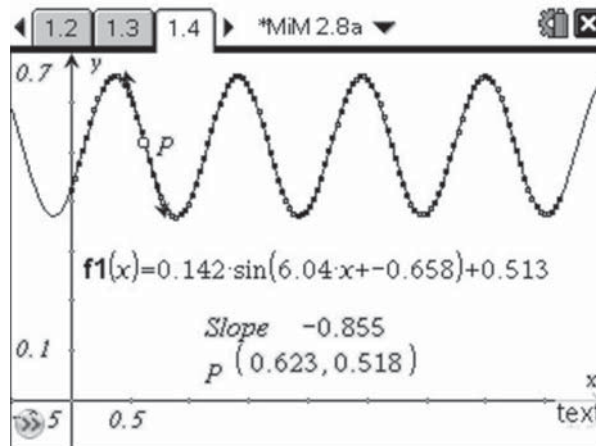

fig_61

fig_62

We can also see that the maximum height of Snoopy is around 0.66m, and his lowest is about 0.37m. These are each about 0.14m above and below the **mean value** of 0.51m – this variation of 0.14m is called the **amplitude** of the oscillation. We can examine the function more closely in a Graphs page. **fig_62**

Snoopy's velocity will be given by the slope at each point P on the displacement-time graph – and this varies between positive and negative small values with slopes of zero at the maximum and minimum.
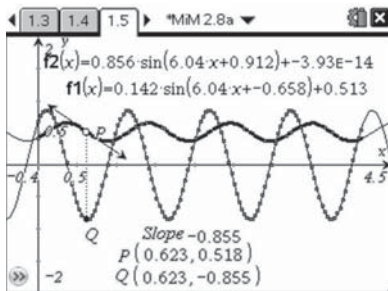
fig_63

fig_64

fig_65

fig_66

In a Lists & Spreadsheets page we can use the function 'nderiv(f1(x),x)|x=a1' in the cell D1 to compute the numerical values of the derivative (i.e. the slope) to the function f1(x) at the value where x equals the contents of cell A1. Filling down the D column with these values gives us the list of numerical velocity values v. Again we can compute a Sinusoidal Regression to fit this data set. **fig_63**

Now we can see that the last, constant, term of f2(x) is really, really tiny (about 4 divided by 100 million million) and can be ignored. So the mean value of the velocity appears to be zero. What can you say about its period, frequency and amplitude? **fig_64**

Also the maxima and minima of the displacement correspond to the zeros of the velocity. To what do the maxima and minima of the velocity correspond on Snoopy's trips up and down? So now we have quite a different kettle of fish from our previous sorts of motion. The slopes of the velocity-time graph are also continually changing. So to study the acceleration we can use the same technique of numerical differentiation to compute approximate accelerations at each interval. We can use the same formula as before, but this time on the function f2(x): 'nderiv(f2(x),x)|x=a1'. **fig_66**

We have computed a list of accelerations, a, and a Sinusoidal Regression f3(x) from them. What can you say about the shape of the acceleration graph? What is its mean value? What is its period, frequency and amplitude? How do its maxima, minima and zeroes relate to the shapes of the velocity and displacement graphs? At what point on Snoopy's trip up and down does he experience the greatest upward pull? When does he feel the greatest downwards pull? **fig_65**

So now we have met the main features of what is called 'simple harmonic motion' (SHM). We just need to explain the roles of each of the four numbers a, b, c, d which TI-Nspire computes as the coefficients of the Sinusoidal Regression function y = a*sin(b*x + c) + d. So let's start with the displacement data and try a manual manipulation of the graph y = sin x. Now we have to get to grips with a possibly new piece of mathematics called the radian. This is quite a different way of measuring angles to our familiar system of degrees. In the toolbar File menu there is an entry for Document Settings where you can change the angle measurement between Degrees and Radians. For this problem you will need to select Radians. In a Graphs page select Function Graph and type sin(x) as the definition e.g. for f4(x). Now, just like linear and quadratic graphs, this basic sine graph can be manually manipulated. Before we do so, let's look at some of its features. You can use Graph Trace to drag a point along it and read off values, such as the maximum. You can also find the Intersection Point(s) between the graph and the x-axis. The first crest of a wave appears at (1.57, 1) and the next zero of the function at (3.14, 0). Do these numbers look familiar?
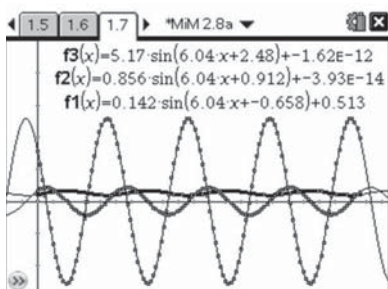
Now we are used to sin(x) being maximum (=1) when x is 90° and to 0 when x is 180°. That, then, gives the clue to how radians are measured. There are π/2 radians in a right-angle, π radians in a straight line and 2π radians in a full circle. So, in radians, the sine function has a period of 2π, a mean value of 0 and amplitude of 1. In order to model Snoopy's displacement data we need a shorter period (greater frequency), a smaller amplitude, a greater mean value and a bit of nudge sideways! In the second of this sequence of screen shots we have flexed the curve to reduce the amplitude leaving the other features unaltered and in the third screen we have raised the curve up to a higher mean value. **fig_67/68**

Zooming in for greater detail we can carry on sliding and flexing until we have a pretty good fit. Now most of the coefficients are self explanatory. The first coefficient, a, controls the amplitude, and the last coefficient, d, controls the mean value. The second coefficient, b, tells us about how long it takes to complete a full cycle. For the simple sine graph, with b = 1, the period (time to repeat) is 2π radians, or about 6.28. Explore the graphs of the functions sin(2x), sin(3x) etc. These "beat" twice as fast, three times as fast etc. So their periods are π radians, 2π/3 radians etc. So in general the period is 2π/b radians, and the frequency is b/(2π). **fig_69/70**

Which just leaves the 'c' coefficient. This is called the **phase shift** or **lag**. Explore the graphs of sin(x+c) for various values of c. What happens when c is π/2, π, 3π/2, 2π? In effect these tell us how far to move the y-axis forward or back to pass through a zero of the function at which the slope is positive (a 'climbing zero'). **fig_71**

So what is it about the experiment which has determined these various characteristics of the motion? To get Snoopy swinging we had to pull him down so that the spring was in tension, and then let him go. So if we had been able to start the video clip at the time of release he would have been at his lowest point (minimum displacement), with a zero velocity, and just about to start moving upwards with maximum positive acceleration. The mean position would have been were Snoopy would have hung from the spring in equilibrium if we had not pushed him downwards. He passes the equilibrium level going upwards with his maximum velocity (and zero acceleration). If the point of release was not too far below the equilibrium level than when he reaches his maximum height the spring still has some tension in it. His velocity now becomes zero and he starts his downward motion with maximum negative acceleration, and so on. In order to change the period of oscillation we need to use a different spring (or elastic rope). It was the great British scientist, Robert Hooke (**http://en.wikipedia.org/wiki/Robert_Hooke**), who first fully investigated this kind of motion in the 17th century.
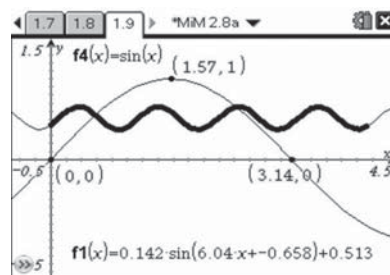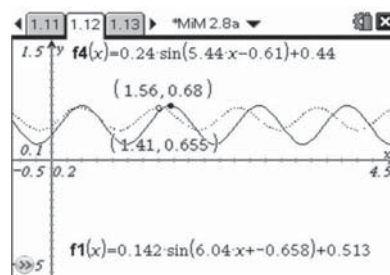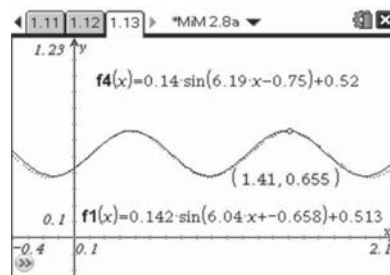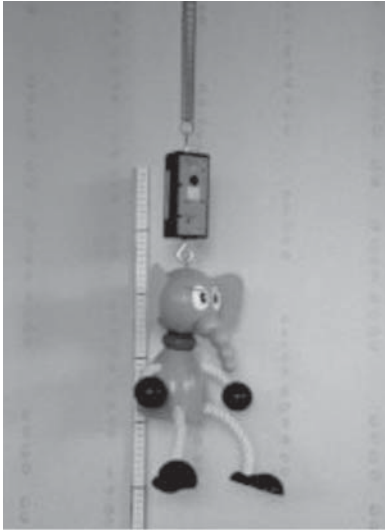
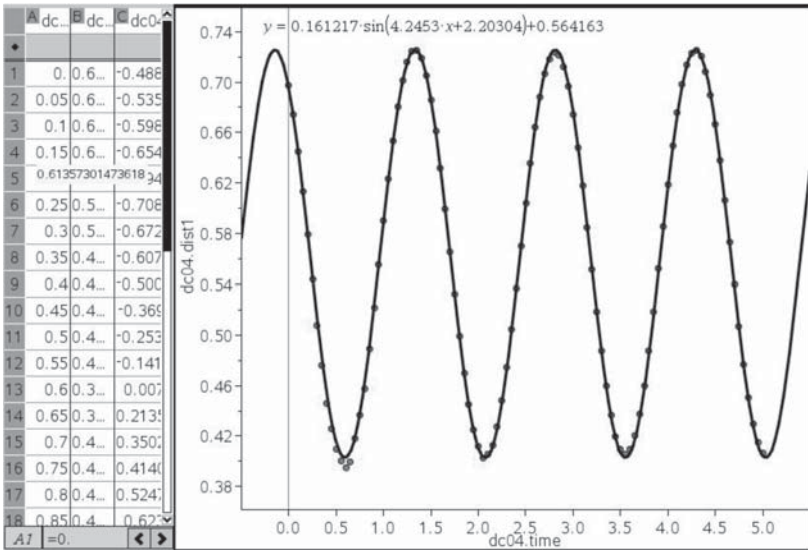fig_67
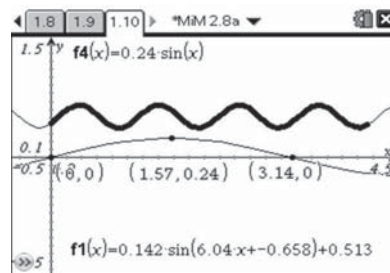
fig_68
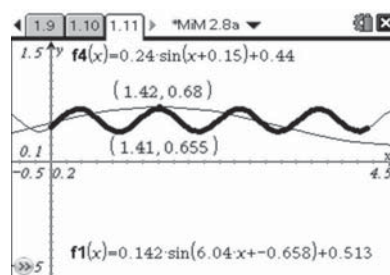
fig_69

fig_70

fig_71

fig_72

In order to see how the parameters change with the kind of spring-mass system we will swap from Snoopy the dog to Ellie the elephant. She is much more massive and her spring is longer, heavier and more tightly wound. She is also capable of carrying an interesting payload! Her photo shows a Vernier wireless dynamic sensor system (WDSS) attached between the spring and her head. This sends data from a force sensor and 3 accelerometers wirelessly to a laptop running the Vernier Logger Pro 3 software. That software can also be set up to control a video camera, or to synchronise with a video clip taken independently of the same motion. We could also set up a CBR2 or Vernier Go!Motion data-logger underneath her to record displacement. So we have a variety of ways we can use to collect data from her motion. **fig_72**

For a change we will use data collected using a CBR2 and a TI-Nspire handheld. This stores data from each capture into pre-named lists such as 'dc01.time' for the first data collected about time. Although TI-Nspire captures time and distance from the CBR2 it also computes its own numerical values for velocity and acceleration. However we will use the same techniques as before to compute our own numerical derivatives. The lists t and d hold the data from our 4th attempt to get Ellie bouncing smoothly! In a Data & Statistics we can see how well a Sinusoidal Regression fits the displacement-time data.
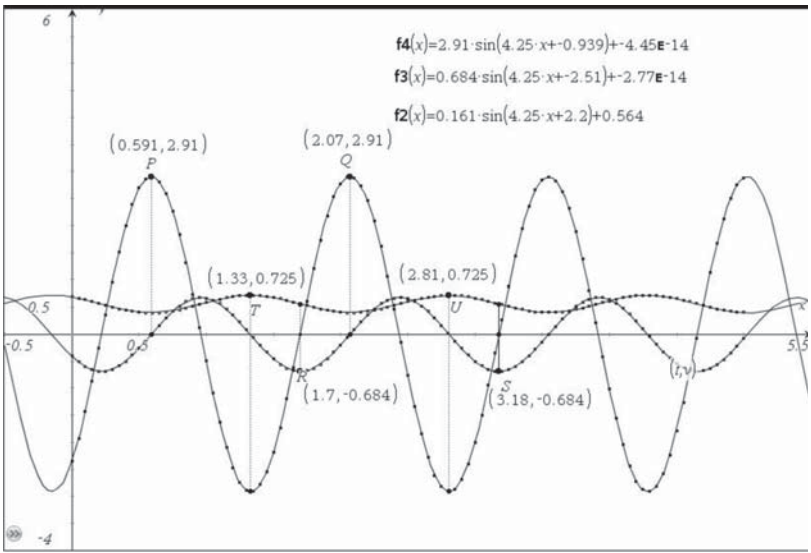
In a Lists & Spreadsheets page we can compute the regression function f2(x) and use it to find the velocities, v, as the numerical derivatives of f2(x) using the formula v:=nderiv(f2(x),x)|x=t, which substitutes in turn each value of t into the formula for x **fig_73**. Similarly we can fit a sinusoidal regression to the v data and store it as function f3(x), ready to find its numerical derivatives as the acceleration a and to find their sinusoidal regression function f4(x). **fig_74**

	A t	B d	C	D v	E	F a	G	H	I	J	K
◆	='dc04.tim	='dc04.dist	=SinReg(	=nderiv(f2(	=SinReg(	=nderiv(f3(	=SinReg(				
1	0.	0.69744	Sinusoi...	-0.402681	Sinusoi...	-2.3462	Sinusoi...				
2	0.05	0.673902	a*sin(b*...	-0.510298	a*sin(b*...	-1.93143	a*sin(b*...				
3	0.1	0.645032	0.1612...	-0.594958	0.6842...	-1.42978	2.90699				
4	0.15	0.613573	4.2453	-0.652854	4.25	-0.863805	4.25				
5	0.2	0.579388	2.20304	-0.68138	-2.51239	-0.25897	-0.9392...				
6	0.25	0.543788	0.5641...	-0.679253	-2.7656...	0.357514	-4.4547...				
7	0.3	0.507497	(0.0032...	-0.646568	(-2.9E-1...	0.957915	(2.473E...				
8	0.35	0.476263		-0.584797		1.51522					
9	0.4	0.445891		-0.496718		2.00437					
10	0.45	0.425753		-0.386293		2.40334					
11	0.5	0.409359		-0.25849		2.69419					
12	0.55	0.400541		-0.119058		2.86385					
13	0.6	0.394743		0.025729		2.90467					
14	0.65	0.399558		0.169359		2.81481					
15	0.7	0.417815		0.30537		2.59833					
16	0.75	0.4364		0.427644		2.26496					
17	0.8	0.457505		0.530679		1.8297					
18	0.85	0.489257		0.609841		1.31212					

D v:=nderiv(f2(x),x)|x=t

fig_73

$$y = 0.161217 \cdot \sin(4.2453 \cdot x + 2.20304) + 0.564163$$

fig_74

$$f4(x) = 2.91 \cdot \sin(4.25 \cdot x + {-}0.939) + {-}4.45\text{E-}14$$

$$f3(x) = 0.684 \cdot \sin(4.25 \cdot x + {-}2.51) + {-}2.77\text{E-}14$$

$$f2(x) = 0.161 \cdot \sin(4.25 \cdot x + 2.2) + 0.564$$

fig_75

Graphing f2(x), f3(x) and f4(x) over the same (time) axis in a Graphs page we can see how the displacements, velocities and accelerations change as Ellie bounces sedately up and down for 5 seconds or so fig_75.

Study the graphs carefully and answer the following questions.

What is the mean value of Ellie's displacement?
What is her lowest height?
What is the amplitude of her oscillations?
What is her maximum velocity?
What is her maximum acceleration?
What is the period of her oscillations?

The points P, Q, R, S, T and U have been marked, and their coordinates shown. For each of these points describe whereabouts Ellie is on that current bounce and what are the time t, displacement d, velocity v and acceleration a at that time.

Below are two representations of the same diagram **fig_76/77**. It shows the scatter plot of velocity v against displacement d. This kind of diagram is known as a **phase plane diagram**. Ideally (with no loss of energy) the points will lie on an ellipse. You have to stretch your imagination to imagine a third, time, axis, coming out of the paper towards you and think about a screw formed by cutting a groove evenly around a cylinder. Each point in space has a coordinate (t, x, y) and we have projected all the points down into the plane where t = 0. The screw thread will have a particular direction. See if you can identify how the points L, U, H, and D relate to each of Ellie's bounces and so determine whether the arrow on the curve should be clockwise or anticlockwise.
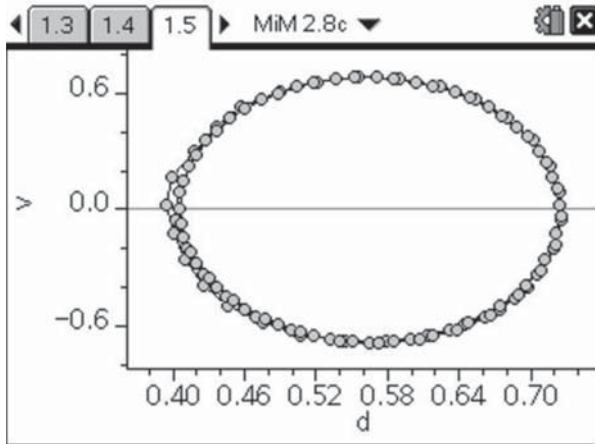

fig_76
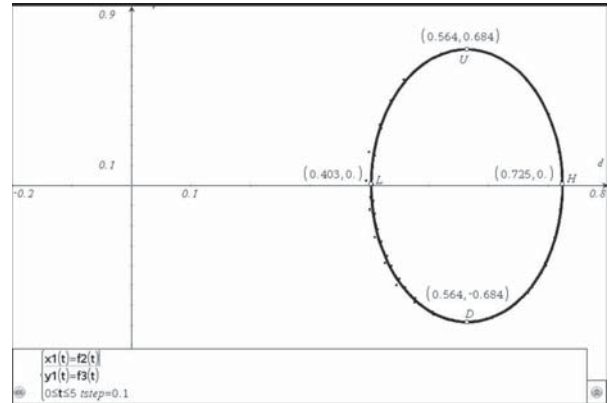
fig_77

Apparatus on a children's playground, or for gymnastics, gives plenty of opportunity for your own study of repeating motion such as swings, see-saws, roundabouts, rings etc. Our next study will be that of a trampoline.

Now, as they say, for something completely different. So far we have looked at kinds of motion for which there are "classical" explanations, once certain simplifying assumptions have been made. For the next couple of examples we will look at more complicated motions, that of a rolling ball and a rocket car.

Chapter

12

Sophie on the trampoline.
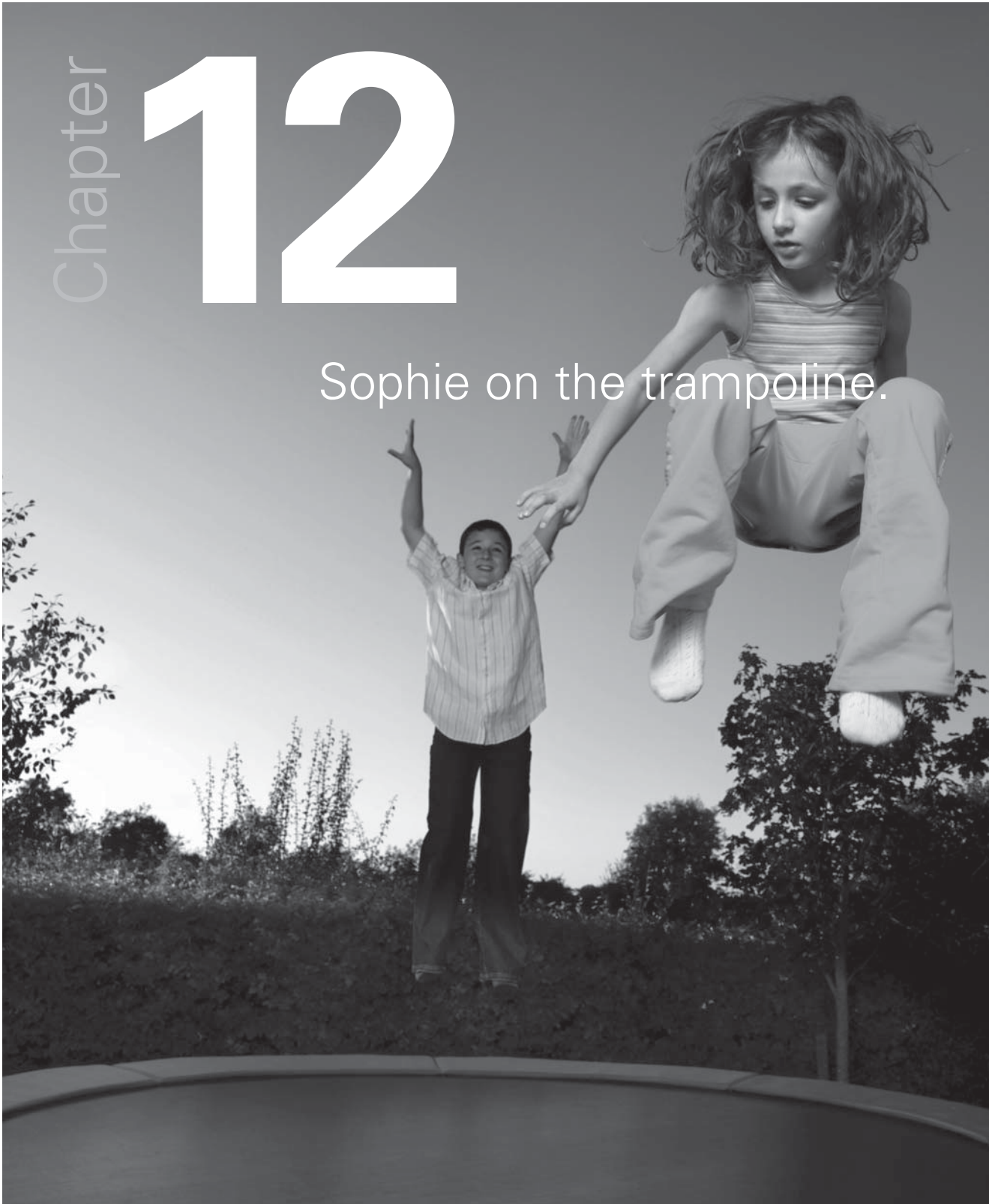
Proud mum Pip has made another video of Sophie in action **fig_78** – not bungee jumping but, more safely bouncing about on a trampoline. Once again we have used Tracker 3 to capture data, taking Sophie's height as 1.5m. You can see the shape of the graph of her height y against time t. Does it look familiar?
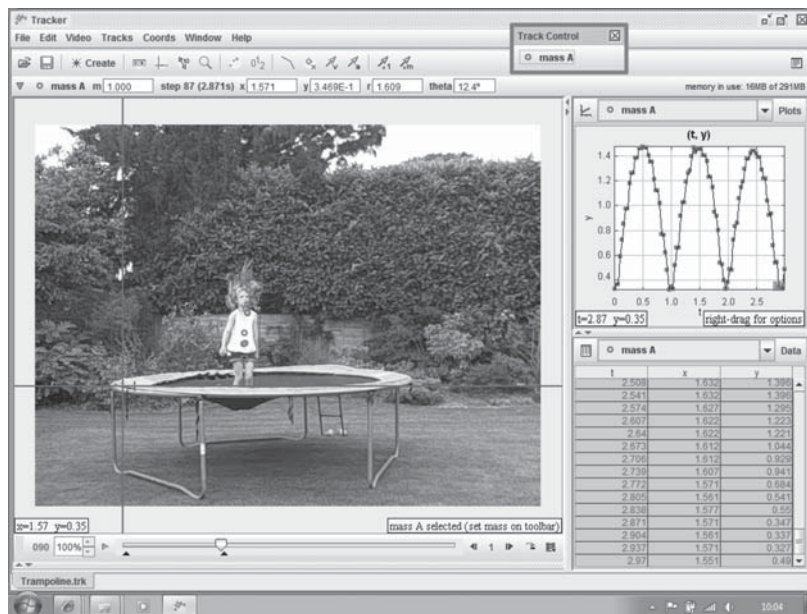

fig_78

Well the data certainly seems to repeat regularly – something we call **periodic** motion. So let's copy and paste into a TI-Nspire document to carry out some analysis. We have stored time in the list t and vertical displacement in the list y. A quick scatter plot and sinusoidal regression reveals that the sine wave is not a very good fit to this data. While repetitive, it can't seem to cope with rounding the corners at the bottom! **fig_79/80**
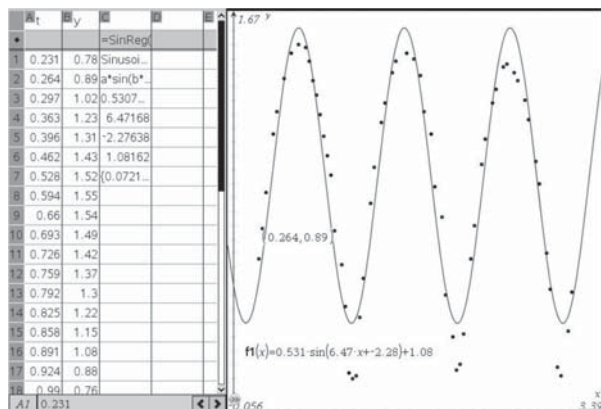

fig_79
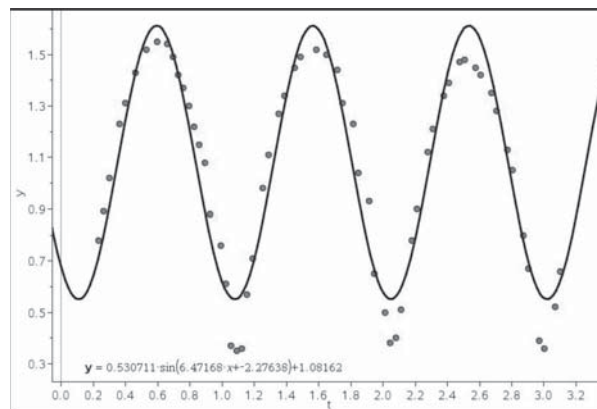
fig_80

Once again we need to use our imagination to describe just how Sophie's movements match with the information on the scatter plots. When she lands on the trampoline she doesn't stop moving, but she starts to stretch the trampoline's membrane. This is like an archer pulling back the string on a bow, or a bungee jumper pulling down an elastic rope or Snoopy/Ellie stretching their springs. So she slows down until she reaches the lowest point at which she is in equilibrium with zero velocity. The trampoline then acts like a catapult to give her an upward acceleration. So the lower part of the motion looks as if it might well be sinusoidal. So let's try the manual manipulation technique to bend and stretch a sine curve. We can certainly get the right sort of amplitude and period, but not the actual shape! Why? **fig_81/82**
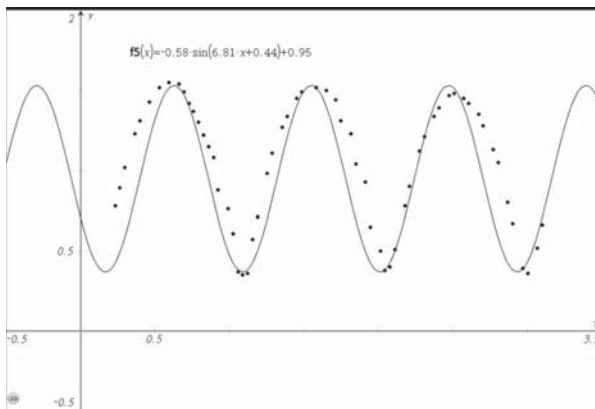

fig_81
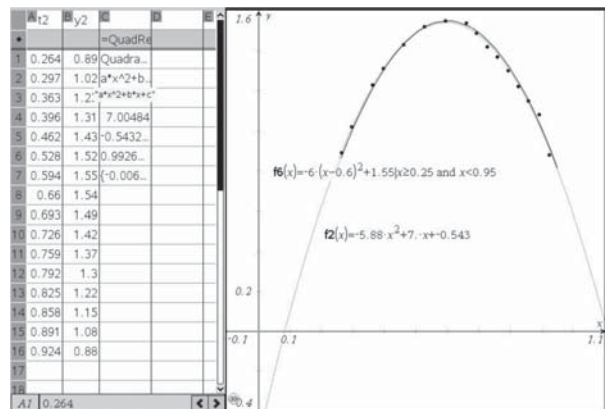
fig_82

Once Sophie's feet have left contact with the trampoline she has become yet another projectile! So she now moves (ignoring air-resistance etc.) with constant acceleration under gravity – and so has linear velocity and quadratic displacement. If we extract a section of data corresponding to her motion clear of the trampoline into lists t2 and y2 we can perform a Quadratic Regression to see if now have a good fit to the upper parts of the motion. Yes – we do! So all we have to do is to do stitch together smoothly lower sections of a sine graph with sideways-shifted versions of the same quadratic function. The screens below give the basic idea. So here the displacement-time graph is a piecewise defined repeating mixture of sections of a sine wave and a parabola. Can you sketch the corresponding velocity-time and acceleration-time graphs? **fig_83/84**
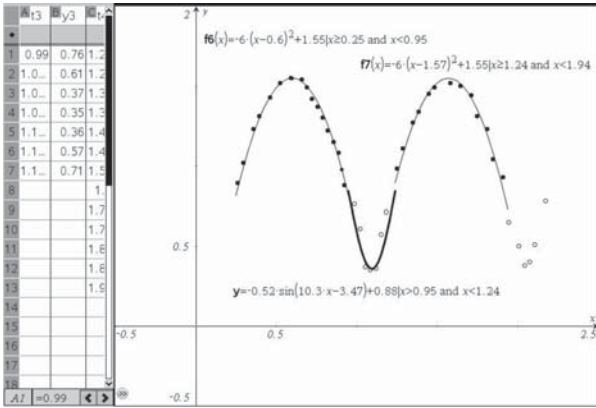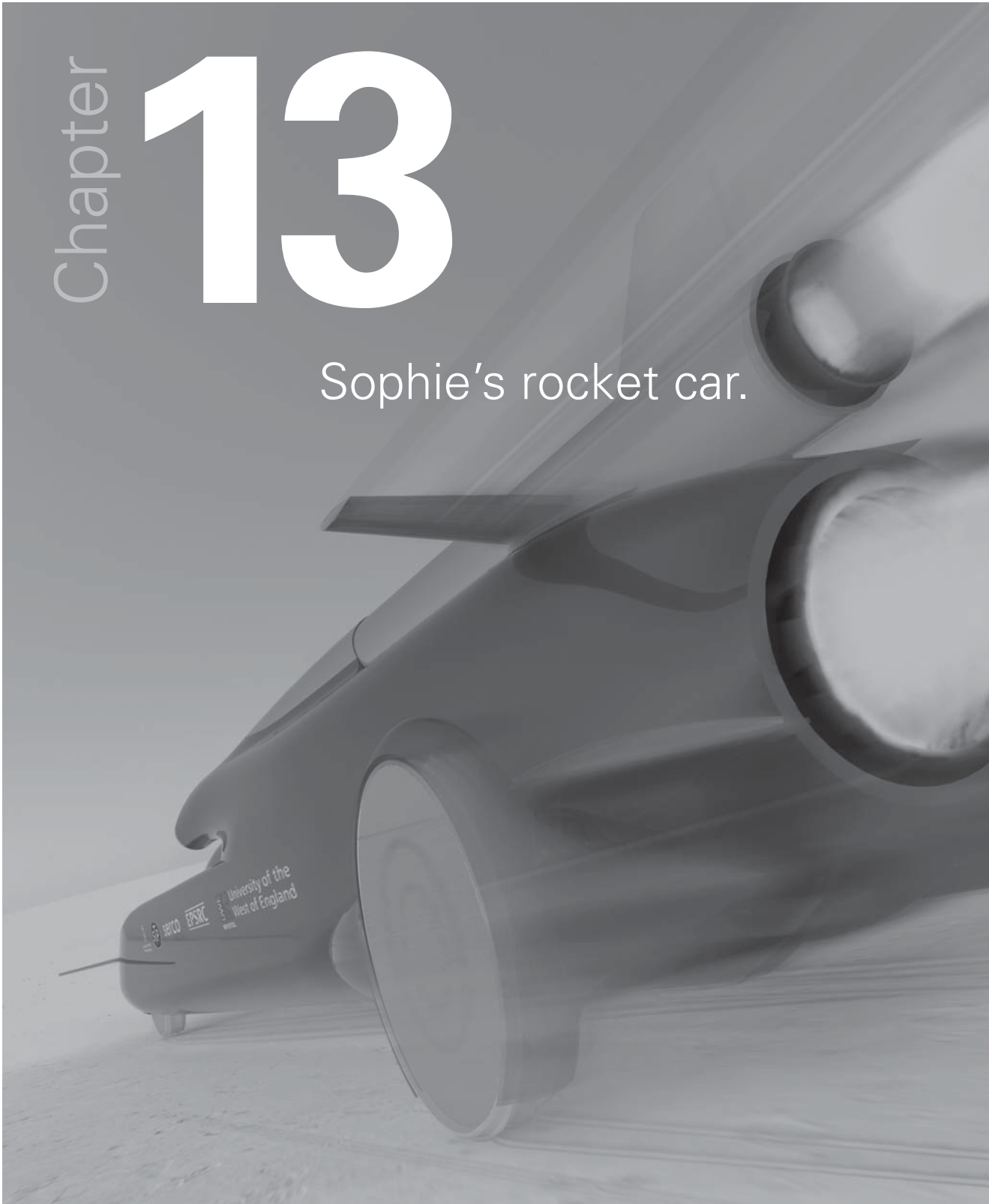
fig_83
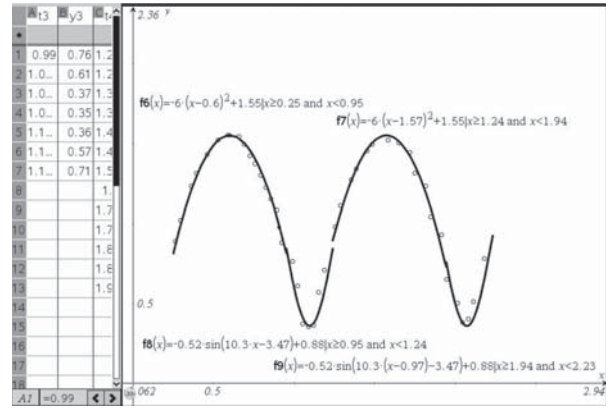
fig_84

See if you can capture some of your own data for this kind of experiment
and produce a smoother set of mathematical results than we have achieved
here so far.

Chapter

13

Sophie's rocket car.

In the January sales, a well-known supermarket chain was selling a range of TopGear toys for half-price. This included a rocket propelled car – which as you can see, Snoopy was hoping to drive – presumably in place of the 'Stig'! **fig_85** Instead Sophie got to half-fill the plastic bottle with water and to put it under considerable pressure with the foot pump – sensibly out of doors! As ever, Pip was there with her high-speed Casio Exilim video camera to capture the car speeding away from an only slightly soaked Sophie **fig_86**. Knowing the car's length as 0.26m we can calibrate the motion in Tracker 3. What sort of shape curve do you think the horizontal displacement against time looks like?
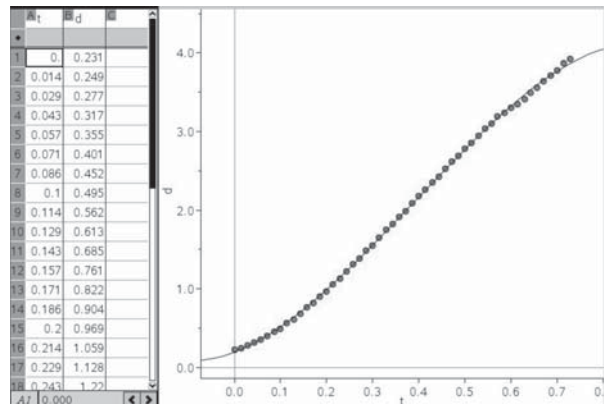

fig_85
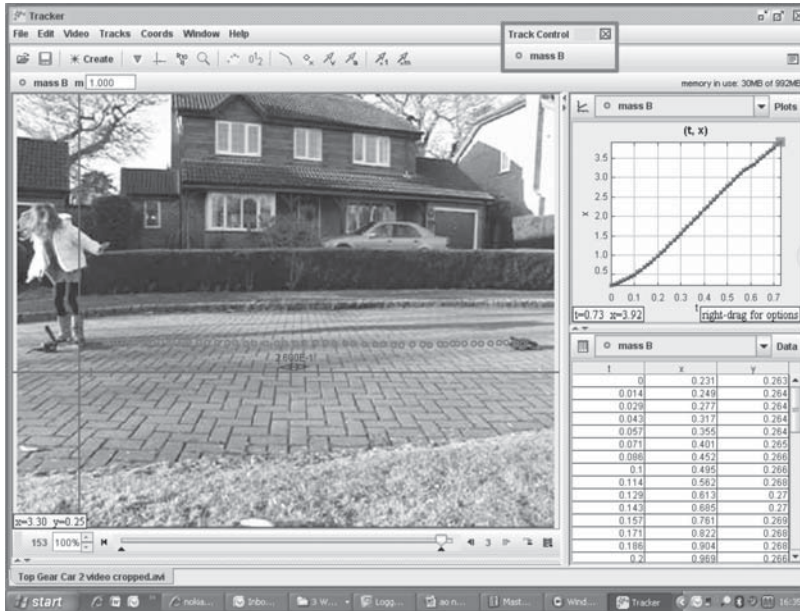
fig_86

Copying and pasting the t and x data into TI-Nspire's lists **fig_87** t and d we can draw a quick scatter plot **fig_88** and test a regression model, such as a cubic – which seems to give a good fit, at least to start with.
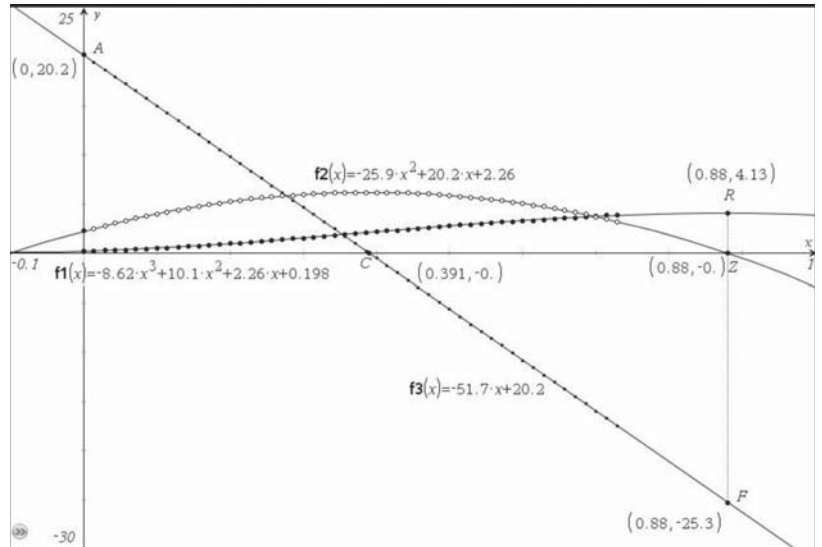

fig_87

fig_88

Once again we can compute the (cubic) regression function as f1(x), find its numerical derivatives and fit a quadratic regression f2(x) to the velocity data, find its numerical derivatives and finally fit a linear regression f3(x) to acceleration data. The result is shown **fig_89**.

fig_89

It suggests our clip from Pip's video didn't quite catch the standing start when the trigger was pressed to launch the car – but it does look as if Snoopy would have felt more than 2g initial acceleration. The frictional forces will at first be overcome by the thrust from the water-jet engine, but as the 'fuel' gets spent so the thrust diminishes until it is all spent. Then the only force acting will be friction (plus air-resistance) which will bring the car to rest.

Can you predict from the graph how long this might take, and how far the car would have travelled? Of course this is quite a complex form of motion because initially at least the mass of the vehicle is changing as the water gets used. So we have a perfect kind of experimental situation for using a variety of data logging. You can use a stop-watch and a tape measure to record the actual duration and range of the trip. You can use video capture to gather data about displacement and time. You could use some form of remote data-capture (like F1 cars' telemetry) to record acceleration, say, and maybe a radar speed gun to record some velocities?

Chapter 14

Rolling a ball – a do-it-yourself experiment.
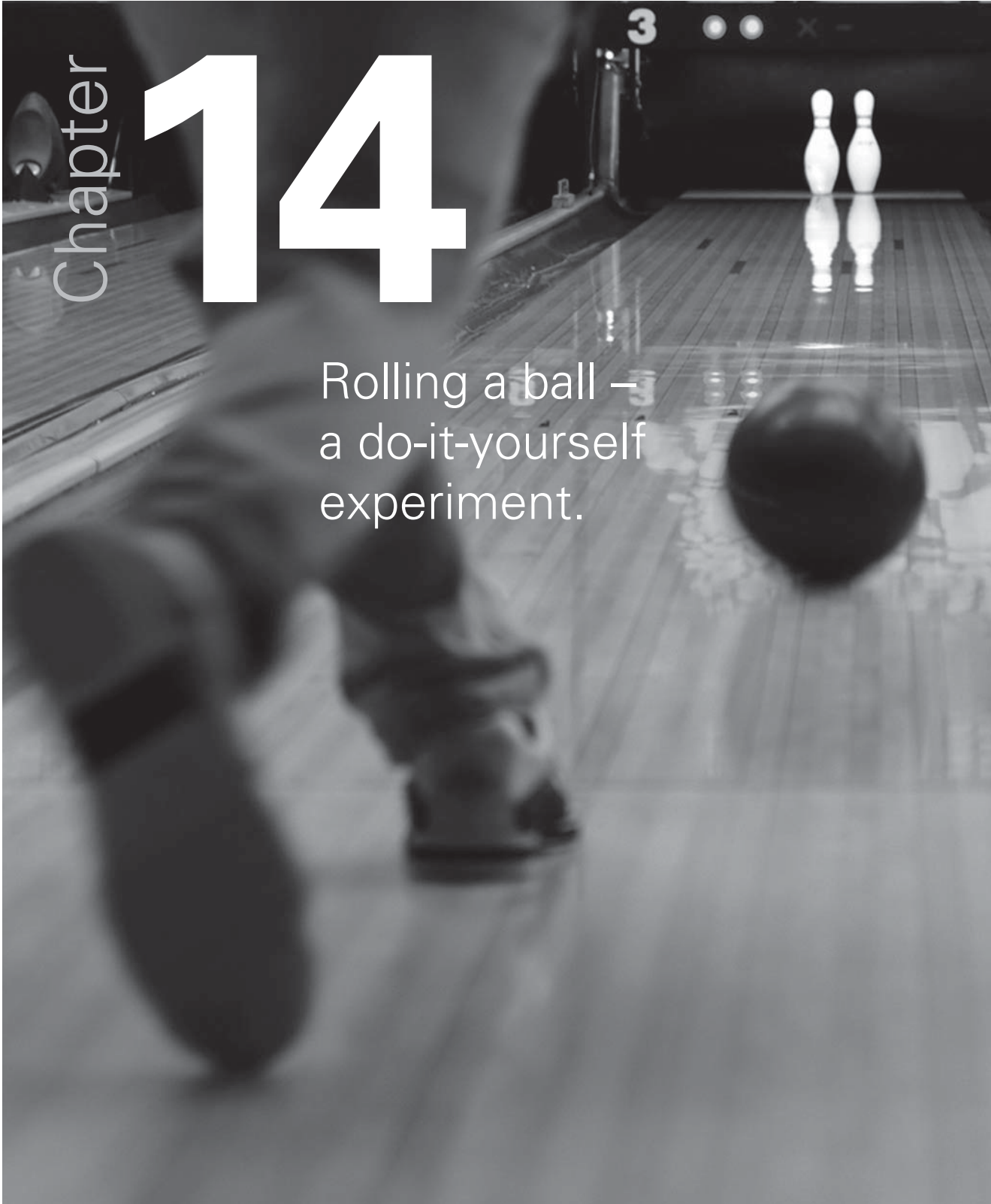
In this very simple example a solid rubber ball is released to roll down an inclined ramp – a section of plastic drain pipe **fig_90**. We can capture the data with a data-logger like the CBR2 **fig_91** and also from a video clip.

The displacement data is recorded by the CBR2 at a sample rate of 50 per second. We can see that a quadratic regression gives a pretty good fit for the data **fig_92**.

fig_90

fig_91

$$y = 0.684383 \cdot x^2 + 0.351014 \cdot x + 0.240211$$
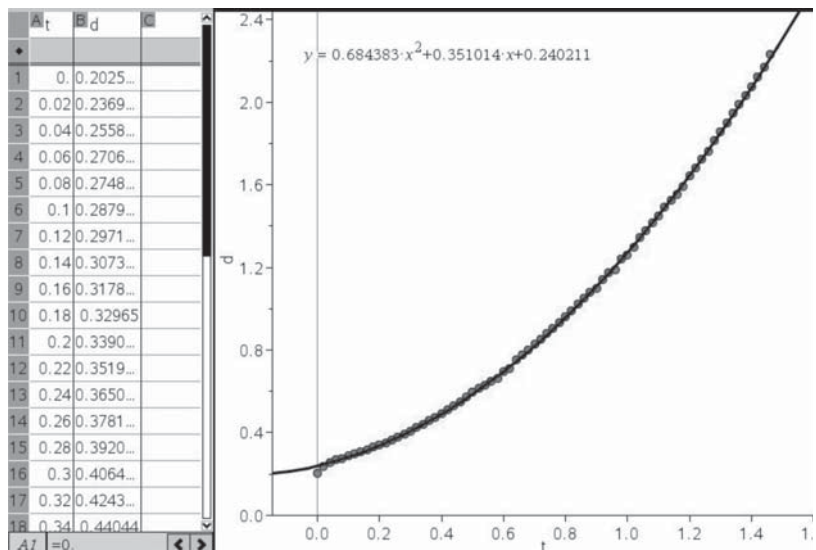
fig_92
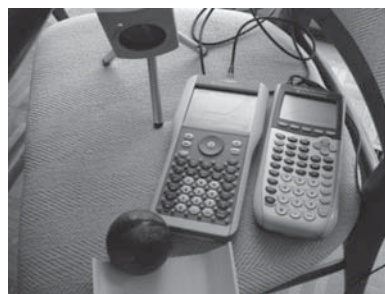
So this a chance for you to put into practice the techniques of this book, using some cleaned-up data from this experiment. The ball was quite close to the CBR2 at the point of release, and so the data may not have been captured cleanly initially. So we can extract data every 0.2s from time 0.2s to time 1.4s to get a better set to work with. These are stored in t1 and d1. **fig_93**

$$f1(x) = 0.7 \cdot x^2 + 0.321 \cdot x + 0.25$$

$(0, 0.25)$
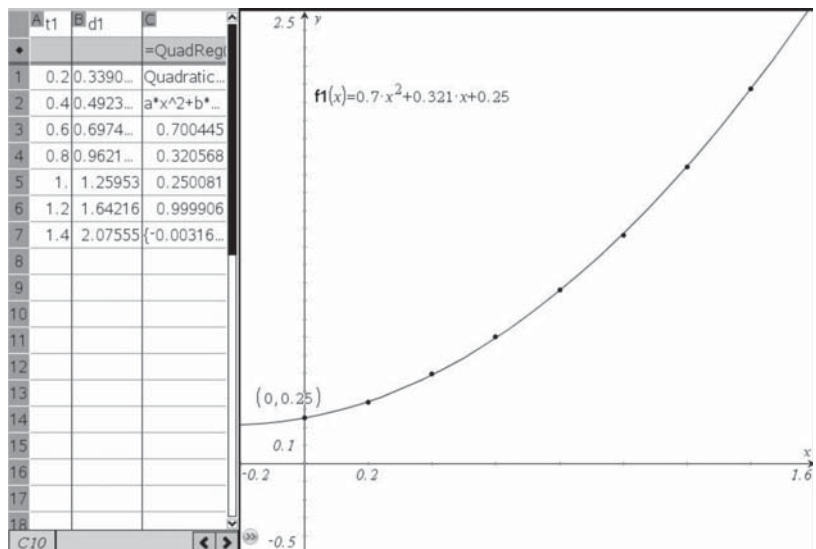
fig_93

You can now copy this data manually into your own Lists & Spreadsheets page and carry out the analysis on the assumption that the displacement is quadratic, hence the velocity is linear and the acceleration constant – perhaps rather like Sophie tobogganing downhill?

For belt and braces we might also record a video clip of the same motion and use Tracker to extract it for comparison.

Here is a video clip taken at 210 fps, but captured at every 5th frame – so effectively a 40 fps movie. **fig_94**
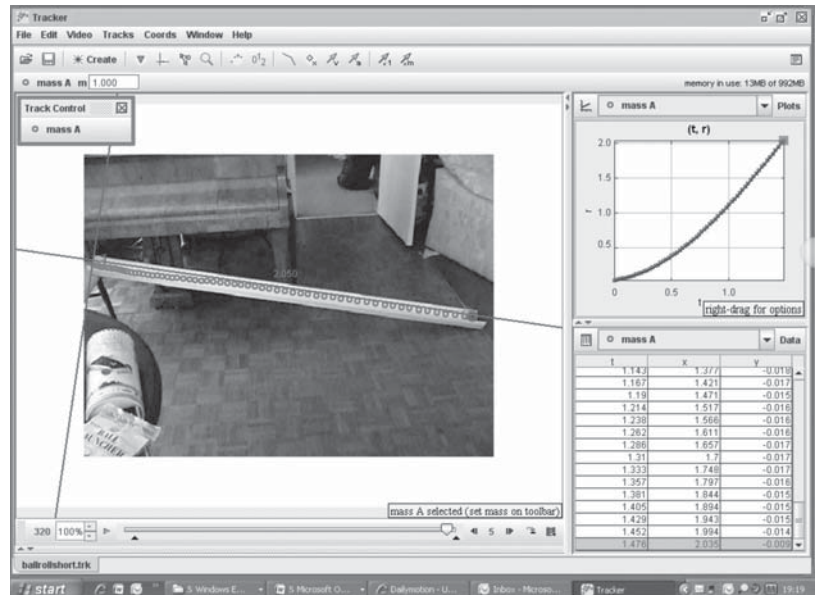
fig_94

In a New Problem the data extracted from the Tracker software video analysis of the ball rolling down the ramp are copied and pasted into the lists t1 and vid. Similarly the data captured from the CBR2 are copied and pasted into the lists t2 and cbr. The lists d1 and d2 are made from vid and cbr by subtracting the first values in each list, so that both measures of distance start with d=0 at t=0. The figure below shows the comparative scatter plots of the two data sets, each with their quadratic regression fit. Here the scatter plots are shown on the same axes, and a quadratic function is manually manipulated to make a good 'by eye' fit. **fig_95/96**

t1	vid	d1 =vid-b	t2	cbr	d2 =cbr-e1
0.	0.034	0.	0.	0.202534	0.
0.024	0.038	0.004	0.02	0.236965	0.034431
0.048	0.049	0.015	0.04	0.255858	0.053324
0.071	0.058	0.024	0.06	0.270678	0.068144
0.095	0.065	0.031	0.08	0.274802	0.072268
0.119	0.074	0.04	0.1	0.287979	0.085445
0.143	0.089	0.055	0.12	0.297169	0.094635
0.167	0.099	0.065	0.14	0.307351	0.104817
0.19	0.115	0.081	0.16	0.317859	0.115325
0.214	0.126	0.092	0.18	0.32965	0.127116
0.238	0.144	0.11	0.2	0.339045	0.136511
0.262	0.161	0.127	0.22	0.351983	0.149449
0.286	0.175	0.141	0.24	0.365074	0.16254
0.31	0.194	0.16	0.26	0.378183	0.175649
0.333	0.214	0.18	0.28	0.392096	0.189562
0.357	0.236	0.202	0.3	0.406488	0.203954
0.381	0.256	0.222	0.32	0.424303	0.221769
0.405	0.278	0.244	0.34	0.44044	0.237906

d2:=cbr−e1

$$y = 0.643864 \cdot x^2 + 0.450448 \cdot x + -0.027791$$

$$y = 0.684383 \cdot x^2 + 0.351014 \cdot x + 0.037677$$

fig_95

$$f1(x) = 0.63 \cdot (x + 0.36)^2 - 0.092$$

CBR data $(t2,d2)$

Video capture $(t1,d1)$

Quadratic flexicurve

fig_96

Of course most schools will have access to proper laboratory equipment such as dynamics tracks and carts, but the great thing about the approaches to data capture considered here is that they are easy and cheap to set up and do not require much technical expertise. The following photograph shows the laptop connected to a webcam on the big tripod and a CBR2 on the small tripod on the chair. The operator can then sit at the laptop and use the yellow folding ruler to nudge the ball down the track while simultaneously starting the data capture. In this instance Logger Pro 3 is being used to synchronise the video capture and the data logging. **fig_97/98/99**

fig_98

fig_97
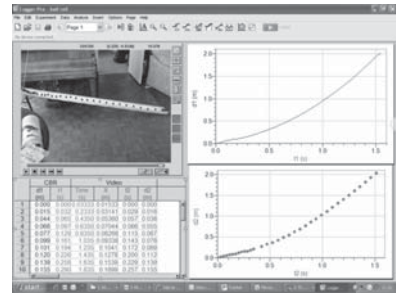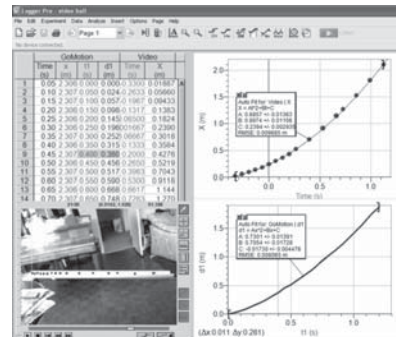
fig_99

Chapter

15

Going for gold –
a Bolt from the blue.

Here are photographs of two recent holders of the world record for the 100m men's sprint. Can you identify them? What have they in common?

fig_100

fig_101

They are Asafa Powell **fig_100** and Usain Bolt **fig_101**, both from Jamaica. Although Powell only finished fifth in the 2004 Olympics in Athens, he returned there the following year and broke the World Record with a time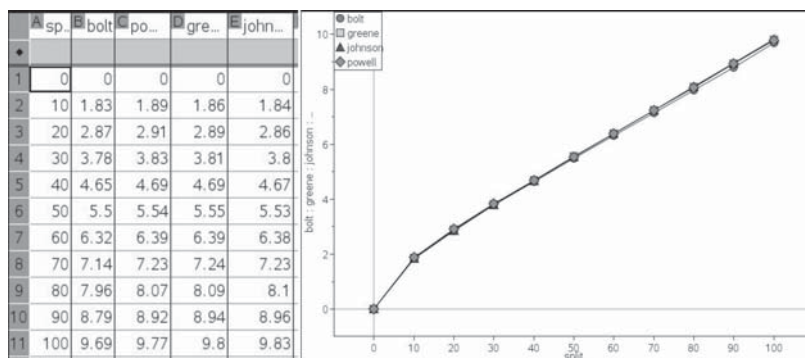 of 9.77s. Usain Bolt of Jamaica won the Olympic Gold Medal in Beijing in 2008 and also broke the world record with a time of 9.69s. In Berlin in 2009 he went on to reduce this to 9.58s. So just how fast might the World record be when the games come to London in 2012? In order to model the speed and acceleration of a sprinter we need more data than just the overall time. With a little hunting on the Web you can find so-called "split" times for how long they took to cover 10m, 20m, ..., 90m and 100m. These are gathered together in the table **fig_102**, together with values for Carl Lewis in 1991 and Maurice Greene in 1997.

|	A sp..	B bolt	C po...	D gre...	E john...
1	0	0	0	0	0
2	10	1.83	1.89	1.86	1.84
3	20	2.87	2.91	2.89	2.86
4	30	3.78	3.83	3.81	3.8
5	40	4.65	4.69	4.69	4.67
6	50	5.5	5.54	5.55	5.53
7	60	6.32	6.39	6.39	6.38
8	70	7.14	7.23	7.24	7.23
9	80	7.96	8.07	8.09	8.1
10	90	8.79	8.92	8.94	8.96
11	100	9.69	9.77	9.8	9.83

fig_102

Their average speed in each case was a little over 10m^{-1}. How fast is this in kh^{-1} and in mh^{-1}?

It turns out the typical reaction time for an international athlete is about 0.13s i.e. the delay between the starting gun firing and the athlete responding. So they actually run a bit faster than the times in the table suggest. They also have to be still at the start, so their initial speed is 0 ms^{-1}. Their maximum speed must also be rather greater than the average. When do you think the speed will be a maximum? What happens after that?

We have drawn scatter plots of the split times for the four athletes, and they all show the same pattern – with the steepest rise happening just about the start, but quickly settling down to a fairly constant speed. We can find the average speed in each interval by finding the difference in times at the end and start of each interval. These will be average times in the interval and so our first guess is they will occur at the middle of each interval. Can you set up a spreadsheet for this, e.g. using Bolt's data? How about a scatter plot for the average speed? **fig_103/104**

|	A split	B bolt	C tgap	D midt	E avvel	F kph	G mph	H zer
◆							=0.625*k	
1	0	0	0	0	0	0	0.	0
2	10	1.83	1.83	0.915	5.46448	19.67...	12.2951	0
3	20	2.87	1.04	2.35	9.61538	34.61...	21.6346	0
4	30	3.78	0.91	3.325	10.989	39.56...	24.7253	0
5	40	4.65	0.87	4.215	11.4943	41.37...	25.8621	0
6	50	5.5	0.85	5.075	11.7647	42.35...	26.4706	0
7	60	6.32	0.82	5.91	12.1951	43.90...	27.439	0
8	70	7.14	0.82	6.73	12.1951	43.90...	27.439	0
9	80	7.96	0.82	7.55	12.1951	43.90...	27.439	0
10	90	8.79	0.83	8.375	12.0482	43.37...	27.1084	0
11	100	9.69	0.9	9.24	11.1111	40.	25.	0

fig_103
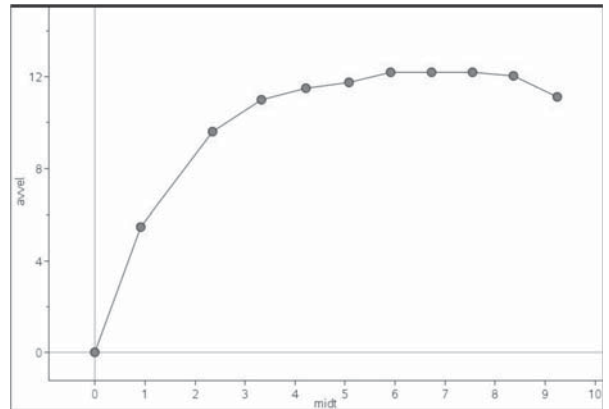
fig_104

Clearly the speed (aka **velocity**) is neither constant nor linear throughout the sprint. So we could try fitting a quadratic regression. But this is not a very good fit to most of the data points, and it doesn't pass through (0,0) – a rather important failure! **fig_105**
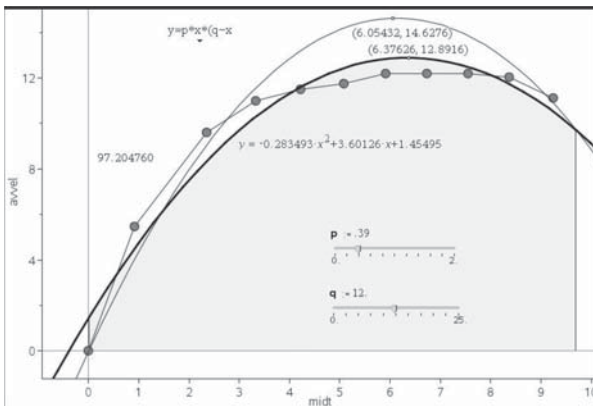
fig_105
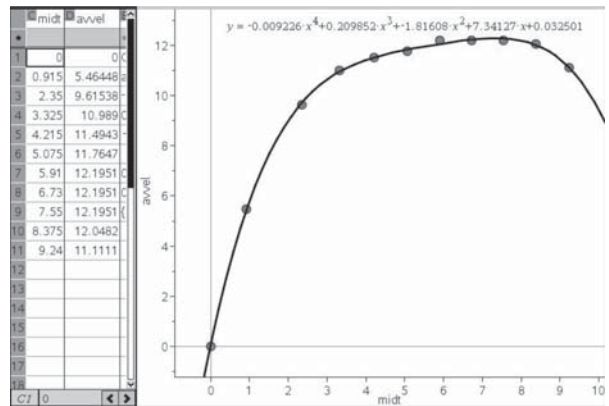
fig_106

We could create our own designer version of a quadratic which always passes through (0, 0):

y = p*x*(q-x). We could set up sliders to find values of p and q so that it the graph passes through the end-points from our table. But this is a worse fit for most of the points!

As we have seen, the velocity-time graph is a particular important one from the modelling viewpoint. The slope at any point tells you about the rate at which the velocity is changing – i.e. the **acceleration** there. The area under the graph between any two times tells you about the distance covered in that time. We can see that the area under the regression quadratic between t=0 and t=9.69 is approximately 97.2m, rather less than the 100m required!

We might conjecture that Bolt runs rather like Sophie's rocket car! He has got himself into the peak of condition and when he bursts off from the starting block he will have zero displacement, zero velocity and maximum acceleration. As he runs he burns up energy (fuel) and cannot maintain a constant acceleration, so the slope of his acceleration-time graph won't stay constant but will be negative – unless, that is, he can find some reserve of energy to put in a 'kick' in the way Oxford and Cambridge try to do during the Boat Race. But if we try a linear model for the acceleration we will just get another quadratic for the velocity. We could try either a piecewise function e.g. two different sections of line graphs – but these can't be smoothly joined – or a higher order polynomial, such as a cubic. **fig_106**

The great advantage of a cubic function for modelling is that it can have a 'wiggle' i.e. a point of inflection as well as possible maxima and minima. A cubic is determined by just four points. So let's try a simple cubic function for acceleration and see if we can push it forward to realistic models for velocity and acceleration? If we had a cubic acceleration model we would expect a quartic (degree 4) velocity model, so let's start with fitting a Quartic Regression to the data for the average velocities in the splits. Pretty good, eh?

We could now find the slopes at some or all of the data points to find the accelerations, and then perform a Cubic Regression on them. If we know a little about derivatives then we could use nderiv as before, but we could also use some calculus directly to find the cubic's coefficients. If $f1(x) = ax^3 + bx^2 + cx + d$ is the velocity model, then its derivative $f2(x) = 3ax^2 + 2bx + c$ is the acceleration model. Also we could integrate it to find $f3(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx$ as the displacement model (no constant of integration required as the initial displacement is zero). **fig_107**

$f1(x)$	$-0.009226 \cdot x^4 + 0.209852 \cdot x^3 - 1.81608 \cdot x^2 + 7.34127 \cdot x + 0.03250$
$\frac{d}{dx}(f1(x))$	$-0.036906 \cdot x^3 + 0.629557 \cdot x^2 - 3.63217 \cdot x + 7.3412$
$f2(x) := -0.036905724384823 \cdot x^3 + 0.62955728602749 \cdot x^2 - 3.6321683400534 \cdot x + 7.34126984468$	Done
$\int (f1(x))dx$	$-0.001845 \cdot x^5 + 0.052463 \cdot x^4 - 0.605361 \cdot x^3 + 3.67063 \cdot x^2 + 0.032501$
$f3(x) := -0.0018452862192412 \cdot x^5 + 0.052463107168958 \cdot x^4 - 0.60536139000889 \cdot x^3 + 3.6706349223423 \cdot x^2 + 0.0$	
	Done

fig_107

If you have the CAS (computer algebra system) version of TI-Nspire it will perform these manipulations for you. The screens below show the graphs of the resulting functions. The quartic f1(x) is a good model for the average velocities. Its derivative f2(x) is a cubic which is a good model for the acceleration. Its integral f3(x) is a quintic (degree 5) which is a good model for the displacement. But what does this tell us about the run? **fig_108/109**
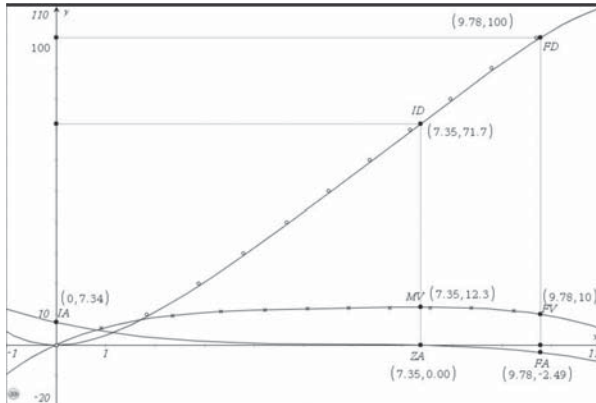
fig_108
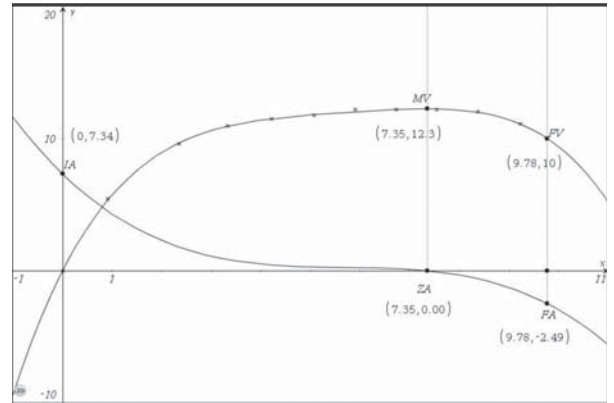
fig_109

But we cannot just leave it like that – we need to interpret what we have done and to see if it is realistic. The first thing to note is that Bolt seems to have set off with an acceleration of over 7ms^{-2}. All the time his acceleration is positive he is increasing his velocity. At the point ZA (zero acceleration) after 7.35s he reached his maximum speed MV of 12.3 ms^{-1}. But then, when we think most sprinters would now continue at this maximum speed until the finishing line, we see that he slows up, since his acceleration becomes negative. His final velocity, at FV, is 9.78 ms^{-1}, meaning he lost about 20% of his peak velocity in the last 2.4s of his run. How can we account for this? Just go back to the photo at the start of this section and look at the margin of Bolt's lead towards the finish. Checking with eye-witness reports this is just what Bolt achieved – a new World Record while slowing down! Clearly it was the Gold medal, rather than the World Record, which was uppermost in Bolt's mind. Notice that we have not made any scientific attempt here to explain the mechanics of sprinting.

The trouble is that sprinting is a very complex form of movement – nothing is moving steadily! The legs pump and down, and hit the track around 40-45 times in the 100m. The arms swing forward and back, while the body arches forward towards the finish. So a mechanical analysis is extremely hard. It would help to get greater amounts of data – either with many more split times, or using several high speed video cameras or with the athlete carrying accelerometers. Maybe a police radar speed gun would also be useful. Even tracking data from video clips is tricky because of the difficulty of finding a steady and visible part of the body to track. So here is a STEM challenge to your school. Can you set up an experiment to get better data for 100m sprinting – particularly in the first 2 seconds, and can you use it to make a better model than we have done here? It would be a good idea for someone with a bicycle and speedometer first to run over the course at a steady 10ms^{-1} so you have a reference reading.

STEM activities with TI-Nspire

The coming decade will see an increasing need for a flexible work force possessing a wide range of skills in Science, Technology, Engineering and Mathematics (STEM) in order to meet the needs of the new high level industries and to be able to produce technologically complex products. The booklets in this series were developed with this need in mind. They aim to provide stimulating activities which link aspects of STEM and at the same time encourage the use of technology and an awareness of its potential.

Why use TI-Nspire for STEM subjects?

TI-Nspire provides a learning platform which dynamically links a variety of ICT applications including documents, graphs, geometry, statistics, spreadsheets, data logging and a calculator. This dynamic linking assists students in making connections not only between different areas of mathematics but also with other areas of the curriculum and STEM subjects in particular.

There is now a wealth of data available on the Internet in a variety of forms which can be copied directly into TI-Nspire. Data can also be captured by linking to a variety of data logging devices such as motion sensors or by recording manually or using video analysis. It is then possible to manipulate, display and analyse this data in a variety of ways or to try to model the data using a variety of data handling and function plotting tools.

About the booklets

All the activities contain some background scientific or other information together with links to appropriate websites. Many of the activities are suitable for a range of ages and aptitudes with more challenging ideas being suggested as extension activities. Further information and TI-Nspire files for the activities can be found at **www.nspiringlearning.org.uk.**

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Introduction - contains a brief introduction and instructions for getting started using some of the features of TI-Nspire that are used frequently in the other booklets.

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