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THE ROLE AND USES OF TECHNOLOGIES FOR THE TEACHING OF ALGEBRA AND CALCULUS

Ideas Discussed at PME over the Last 30 Years

INTRODUCTION

In the history of humankind, many "representational infrastructures" (Kaput, Noss & Hoyles, 2002) were introduced, as written language, number systems, computation systems, algebraic notations, which gave the possibility to register, transfer and record various kinds of information, and also to support the capacities of the human brain. The notion of an automatic computing machine that precedes the modern computer is not new: Leibniz was searching for such a kind of tool, being aware of the fact that "not only that choice of notation system was critically important to what one could achieve with the system, but also and more specifically, that a well-chosen syntax for operations on the notation system could support ease of symbolic computation" (Kaput et al., 2002). Technology can be seen not only as the last powerful representational infrastructure introduced by humankind to present thoughts, to communicate and to support reasoning and computation, but it can also be seen as an infrastructure that supports at least two developments: Human participation is no longer required for the execution of a process, and the access to the symbolism is no longer restricted to a privileged minority of people, as it was in the past. The first development caused the incoming of new kinds of employment, and the death of others. The second is responsible for a general democratisation of access to knowledge, particularly in the mathematics and science disciplines.

Following Kaput et al. (2002), we can say: "The extent to which a medium becomes infrastructural is the extent to which it passes as unnoticed". Representational forms are often transparent to the expert user: Musicians do not think about musical notation when they play an instrument, any more than expert mathematicians do. Transparency can be reached through using the instrument, but also the evolution of technology can help this process. Since the first technological instruments were introduced at school (more or less thirty years ago), more and more people have gained access to them, because of the creation of new interfaces that mediate our knowledge in using them (in terms of operating systems, programming languages, and so on). So, not only the new technologies can be seen

A. Gutiérrez, P. Boero (eds.), Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future, 237–273.

as a more democratic representational infrastructure, with respect to the old ones, but within the new technologies themselves. In their evolution in the last thirty years, we have seen a democratisation process in the sense of an increasing number of users. On the one hand, there has been a reduction of the competencies needed to use technologies; on the other it has also reduced the need to make sense of how computational systems do what they do.

So if the machines can perform calculations, what is left of mathematics? Almost everything. Machines cannot do argumentations, reasoning, conjectures, proofs (not in the sense of automatic proof, but justifying the passages) and so on. These are peculiar to the human capacity of reasoning: "The devolution of processing power to the computer has generated the need for a new intellectual infrastructure; people need to represent for themselves how thinks work, what makes systems fail and what would be needed to correct them" (Kaput et al., 2002).

Over the last thirty years, technology has shaped the way algebra is perceived. Algebra continues to be seen as an extension to arithmetic but technology is allowing students to explore the symbolic language as a computational tool and as an entry point to the major concepts in calculus. At the same time, algebra's symbol system is being linked more powerfully to the tabular, geometric and graphical contexts. Central concepts in algebra, like those of variable and function, can be treated dynamically in contrast to conventional paper and pencil technology where they are constrained to a static existence. Similarly, key ideas in calculus, such as limit, derivative and integral, benefit from dynamic representation that digital technology affords. Allied to the provision of a more dynamic representation, new technology promises the potential for a more interactive experience. Perhaps as a consequence, research interest and teaching focus have begun to emphasise the construction of meanings more than symbolic manipulation.

How technology is used inevitably depends upon the task. For example, to solve mathematical problems, technology is often used to take care of the calculations to simplify or verify the activities. In a more complex task, where the pedagogic aspiration may be turned more towards sense-making activity, technology may be used to explore, to conjecture, and to test conjectures, to validate a statement just found and to express the mathematical idea in a formal manner.

In fact the reader at this stage may begin to glimpse the complexity of issues that will pervade our review of the impact of technology on algebraic and analytical thinking. The complexity is not simple to deal with. In fact, as Hershkowitz and Kieran (2001) point out: "In designing as well as in studying a classroom learning activity in a computerized mathematics learning environment, one should consider contextual factors of various origins, like: (a) The mathematical content to be learned and its epistemological structure; (b) The learners, their mathematical knowledge culture, and the history with which they started the researched activity; (c) The classroom culture and norms, the role of the teacher, the learning organization –in small groups or individually–, etc.; and (d) The potential 'contribution' of the computerized tool".

Technology re-awakens us to the complexity that fundamentally underlies the teaching and learning of mathematics; in effect age-old questions are re-energised: "Regarding educational goals, appropriate pedagogical strategies, and underlying beliefs about the nature of the subject matter, the nature of learners and learning, and the relation between knowledge and knower", and their implementation "also forces reconsideration of traditional questions about control and the social structure of classrooms and organizational structure of schools" (Kaput, 1992).

In this chapter, we have decided to map out this complexity by focussing in turn on each of three domains. In the next section, we consider how PME researchers have studied the use of technology with respect to expressions and variables. In the second section, we set out the research on algebraic notion of function. The third of these sections considers technology with respect to calculus concepts.

TECHNOLOGY IN THE TEACHING AND LEARNING OF EXPRESSIONS AND VARIABLES

As we trace the history of PME research on the teaching and learning of expressions and variables with technologies over the past thirty years, we note different kinds of uses or supports as offered by different tools. Using technology to support the teaching of a mathematical topic demands a transformation, sometimes in the mathematics itself but often in the pedagogic stance of the teacher. The mathematical notations traditionally evolved in the context of static, inert media, whereas the advent of technology brought changes in the perspectives from which a concept can be seen, progressively introducing interactivity and dynamicity. In this respect, the introduction of technology to the teaching of expressions and variables has been linked to reforms in the teaching of the subject generally. The nature of the transformation will depend upon the topic and so we find a connection between the topic and the type of tool used. As a consequence, "the question of whether a child can learn and do more mathematics with a computer (or other forms of electronic technology, including calculators and various video systems) versus traditional media is moot, not worth proving"; and "the real questions needing investigation concern the circumstances where each is appropriate" (Kaput, 1992).

From this historical review, we note a significant change occurring at the end of the '80s, when research started to pay attention to the relevance of multiple representations to the teaching of algebra. Prior to that there had been a major interest in the use of programming to approach specific contextual knowledge, for example that related to the concept of variable. During those years a flourishing use of languages such as Logo, Pascal, Basic, and others in teaching expressions and variables influenced research in the field. Towards the end of the decade, research interest in programming as a medium for learning about expressions and variables waned. It is not clear whether this is mostly a matter of fashion or whether somehow the complexity of learning the particulars of the language was seen as counter-productive to the intended development of general cognitive and thinking skills through the programming activity. This feature of 'less or nonimmediacy' is one of the major difference between programming and microworlds, where the students' actions on objects is more direct than in a programming environment. The increasing development and experimentation of these kind of microworld supported the bridge of the gap between manipulation skills and abstract reasoning with algebraic symbols.

The previous perspective has strongly characterised research on the teaching and learning of expressions and variables over the last decade, when new technology (as for example computer algebra systems, and symbolic-graphic calculators) entered the scene. Studies have mainly focused on investigating students' appreciation of formal algebraic notations, generalisation and abstraction processes, and meaningful construction of symbolic language for expressing mathematical ideas. The uses of technology varied from graphing calculators supporting only formal algebraic notations, to non-standard algebraic notations of spreadsheets, and to microworlds specifically designed to learn different aspects of algebra.

Programming and the concept of variable

Toward the end of '80s the first investigations of the use of *programming* in teaching and learning began. Languages, such as Logo, Pascal, and LSE became popular in educational research as a means to analyse the difficulties encountered and approaches used by students in acquiring the concept of *variable*. In fact, the solution of a programming problem is not a result but a procedure to be represented by the subject as a function operating on data; this representation entails a consideration of the data as variable. However, the use of programming languages is not an easy matter and needs some further reflection. First, it is not possible to study the concept of variable independently from the programming languages and the domain of problems to solve; for example, operating on numbers or characters or graphical objects has not the same meaning for the subjects (Samurçay, 1985). In addition, variables can have different functional status in that their values can be in users' or in programmer's control (as explicit inputs and outputs of a problem or as variables only necessary for its solution). Two consequential effects have to do with the design of classroom activities: On the one hand, the need for a definition of the conceptual field about which the didactic experiences are to be organised; on the other, an attention to the nature of objects that can be manipulated with a given language. Samurçay (ibid.) set out the main aspects related to a teaching approach using programming. 8-9 year-old children who used Logo appeared confused between objects and the procedures defining them, which seemed to be a specificity of learning programming. For 16-17 year-old college students using Pascal and LSE, troubles were in conversion of their algebraic description into a procedural description.

Regarding students already used to programming in Logo, a strong hypothesis was that certain programming experiences could provide students with a conceptual basis for variable, enhancing their work with paper and pencil algebra (Sutherland, 1987). The relevance of integrating different learning environments (paper and pencil, and technology), rather than abandoning one in favour of the other, appeared. Sutherland (ibid.) tested the hypothesis analysing the activity of pairs of students playing a game. The game involved one pupil defining a function and the other pupil predicting the function by trying out a range of inputs. The latter had to define the same function when he/she was convinced that his/her prediction was correct. The pupils then had to establish that both functions were identical in structure although the function and variable names used might be different. Later individual structured interviews showed that students were able to use their Logo derived understanding in an algebra context.

Structures in expressions and equations

The importance of *structure* was also the focus of research in mathematics education concerned with the study of *expressions and equations*. Many of the errors in manipulating an algebraic expression seemed to be due to students' inattention to the expression's structure: Parentheses or conventions for the order of operations (Thompson & Thompson, 1987). A special computer program, called Expression, was developed in order to enable students to manipulate expressions with the constraint that they could act on an expression only through its structure. The program showed both the sequential format of expressions and the form of an expression tree. The statistical analysis of 7^{th} graders' responses to some numeric-transformation and identity derivation problems suggested that when the students internalised the structural constraints they were less likely to commit errors and were more efficient in their solution strategies. This behaviour arose from the fact that pupils could attempt a lot of incorrect transformations of expressions by using the computer, but the program would not carry them out. Even experimentation became natural and beneficial thanks to the availability of different representational systems of an expression.

Concerned with solving equations or producing *equivalent* equations, a real problem for students is recognising and understanding even a simple case of equivalent equations. Recent studies have tried to elaborate on ways in which technology might be helpfully used to overcome such an obstacle. For example, Aczel (1998) claims that the use of a simulation of a balance could improve children's knowledge of equation solving. Other researchers consider the effectiveness of the use of an Interactive Learning package, called The Learning Equation (Norton & Cooper, 2001). They describe students' views about working with the software, concluding that it provides cognitive scaffolding.

Multiple representations

The significance of *multiple representations* and their mutual links rose in the late '80s, when research was beginning to identify specific reasons why algebra is so hard to learn and what the appropriate curricular and pedagogical responses might be (Kaput, 1987). Of course, it is not an easy task since algebra is complex both in its structure and in the multiplicity of its representations. But the representational aspects are essential. Mathematical meaning can be naturally grasped by: Transformations within a particular representation system without reference to another representation; translation across mathematical representation systems; translations between mathematical and non-mathematical representations (e.g. natural language, visual images, etc.); consolidation and reification of actions, procedures and concepts into phenomenological objects, which can serve later as the basis of new actions, procedures and concepts (Kaput, ibid.). As a consequence, meanings are developed within or relative to particular representations. Take the mathematical word "function" as an example. There is no an absolute meaning for it; there is, however, a whole range of meanings depending on the many available representations of functions and correspondences. Think of a function as a transformer of numbers (that is a typical instance of procedural meaning), or as a relation between numbers (which instead is a case of relational meaning). Each of these meanings is then associated to some specific representation, such as: "f(x)=..." for the first example above, and "y=..." for the second example. Furthermore, individuals use representation systems to structure the creation and elaboration of their own mental representations. In light of these perspectives, a central goal of algebra research became to determine how those representational forms are learned and applied by individuals to produce useful mental representations. For what explicitly concerns technology, computer-based models came to make multiple representations available, with the additional feature of serving not simply to display representations but especially to allow for *actions* on those representations (Kaput, ibid.). Here is one of the reasons that the idea of variable had been so difficult to learn: The static nature of the media in which everybody had historically been forced to represent it.

A dynamic view of algebra

Within the previous perspective, a dynamic view of algebra flourished; a lot of software and games were designed to favour it. The dynamicity is a fundamental feature of the new developing media. In fact, as Kaput (1992) stresses "one very important aspect of mathematical thinking is the abstraction of invariance. But, of course, to recognize invariance –to see what stays the same– one must have variation. *Dynamic media inherently make variation easier to achieve*" (emphasis in the original). A particular case was the attempt to improve the conceptual understanding of the *use of letters* in algebraic expressions and equations. For example, software conceived as generic organisers had a wide diffusion. A generic organiser is a "microworld which enables the learner to manipulate examples of a concept. The term "generic" means that the learner's attention is directed at certain aspects of the examples which embody a more abstract concept." (Tall, 1985). It provides an environment, which enables the users to manipulate examples of a specific mathematical concept or a related system of concepts, to aid the learners in the abstraction of the more general concept embodied by the examples (Thomas & Tall, 1988). The software gives an external representation of the abstract concepts and acts in a cybernetic manner, responding in a pre-programmed way to any input by the users. In this way, it enables both teacher and pupil to conjecture what will happen if a certain sequence of operations is set in motion, and then carry out the sequence to see if the prediction is correct. As a result of a long-term study with 11 and 12 year-old algebra novices, Thomas and Tall (1988) outlined that the generic organiser allows for an ideal medium to manipulate visual images. In so doing, it acts as a model for the mental manipulation of mathematical concepts, entailing emphasis on conceptual understanding and use of mental images rather than skill acquisition. Long-term conceptual benefits and a more versatile form of thinking related to the experiences with the computer were evident by the study.

Algebra as a symbol system

Taking into account the difficulty in bridging the gap between algebra as an extended arithmetic and algebra as a formal system of symbols, some researchers began to design novel activities through the use of fresh microworlds that provide access to multiple representations. In Israel, grade 8 students were involved in an experimental work with a computer package, which combines *skill-drills and logical reasoning* by competitive games relative to *substitution in algebraic expressions* (Zehavi, 1988). The positive effects coming from the use of the software were clear thanks to the comparison with a control group and discussion with the teacher in cognitive workshops.

Robust mathematical meaning can also be supported by the relations of the mathematics in a problem with the *relevant situational knowledge* of the problem. Around this belief, some research in mathematics education built up computer animation-based tutors to enhance students' mental representations. One tutoring system, called Animate, was specifically developed to provide students with an improved ability to generate a formal set of algebraic expressions or equations from problems presented in story form (Nathan, 1992). A positive effect of such a tutor came from its interpretive feedback to the students: In fact, they could continue to develop expressions containing conceptual errors, learning to detect and repair them in the process. As a result, students refined the solutions in an iterative way until the situation and the mathematics were seen as mutually consistent. Their competency in interpreting abstract expressions in a situational way improved. On the basis of this analysis, Nathan (ibid.) highlighted that the coupling of the mathematical expressions to a concrete depiction of the situation is necessary.

Meaningful symbolic syntax

Algebra as a symbol system entails a meaningful view of variables, unknowns, and parameters in formulas as well as in expressions and equations. Some recent studies have looked at the cognitive aspects of the abstraction and generalisation processes in learning environments supporting different types of *algebraic notations*. For example, Yerushalmy and Shternberg (1994) compare generalisations of number patterns found by students with a microworld, called

Algebraic Patterns, and with paper and pencil. Algebraic Patterns displays a dynamic numbers' lattice, and provides tools to describe local relations either by one or by many variables, and by functional notations. Arzarello, Bazzini and Chiappini (1995) sketch a theoretical model for analysing students' activities of production and manipulation of algebraic *formulas* using a spreadsheet. They examine iteratively and with increasing detail the relationships between the model and the learning environments. In so doing, they generate a fine-grained description of the features of school activities that support a meaningful learning of algebra. On the other hand, Ainley (1995) addresses the early stages of children's introduction to the use of variables in formal algebraic notation. Her conjecture is that some of the difficulties encountered by children in this area may be accentuated by their lack of appreciation of the purpose, or power, of *formal notation*. Ainley (ibid.) aims at situating the use of formal notation in meaningful contexts. She invoked case study evidence from children working with this approach, using spreadsheet and graphical feedback in problem solutions to suggest links to other areas of cognitive research.

The implementation of tools as media to give powerful visual insights supporting the generation of algebraic meaning, and to bridge the gap between action and expression, is the focus of research by Healy and Hoyles (1996). They studied 12 year-old students' use of spreadsheets and the Mathsticks microworld to examine their visual and symbolic strategies while interacting with these software environments.

A wide study on the teaching and learning of Algebra as a theory has been carried out more recently by some Italian researchers (Mariotti & Cerulli, 2001). They focused on the idea that a technological tool is seen as an instrument of semiotic mediation the teacher can use in order to introduce pupils to a theoretical perspective. The didactic problem considered concerns the ways of realising a theoretical approach to symbolic manipulation. A key-point of the research:

> Is that of stating the 'system of manipulation rules' as a system of axioms of a theory. The nature of the particular environment may foster the evolution of the theoretical meaning of symbolic manipulation. This is not really the approach pupils are accustomed to, on the contrary, Algebra, and in particular symbolic manipulation, are conceived as sets of unrelated 'computing rules', to be memorized and applied. (Mariotti & Cerulli, ibid.)

Within this perspective, a microworld, L'Algebrista, was designed, incorporating the basic theory of algebraic expressions. Algebra theory, as far as it is imbedded in the microworld, is evoked by the expressions and the commands available in L'Algebrista. A significant point of the activity is the fact that L'Algebrista is a symbolic manipulator totally under the user's control: The user can transform expressions on the basis of the commands available; these commands correspond to the fundamental properties of operations, which stand for the axioms of a local theory. As a consequence, the activities in the microworld, which produce a chain of transformations of one expression into another, correspond to a proof of the equivalence of two expressions in that theory. A further issue, which is not explicitly discussed in the research but it is worthy of attention, is that concerning the role of the teacher in the process of evolution of meanings. The researchers assert that the role of the teacher "becomes determinant in a process of de-contextualisation required in order to redefine the role of "buttons", and "new buttons", outside the microworld. In fact, commands must be detached from their context and explicitly referred to mathematical theory. Further investigations into the delicate role played be the teacher are required for a better and clearer description" (Mariotti & Cerulli, ibid.).

CAS and symbolic-graphic calculators

The introduction and diffusion of CAS (Computer Algebra System) and symbolic and graphic calculators to teach elementary algebra mainly occurred over the last decade. As a consequence, research work on their use and implementation at school is still little and can be considered at its early stages. One of the studies carried out with algebra beginners (11-year old Mexican students) investigates the extent to which the use of a graphing calculator can help as a tool (Cedillo, 1997). The study points out that the language of the calculator turns out to be means of expressing general rules governing number patterns, helping children grasp the algebraic code.

Looking specifically at the use of CAS, Drijvers (2001) studied how they can contribute to a higher level of understanding of parameters in algebra. Throughout the analysis of a classroom episode, the research discusses the relationship between machine techniques and mathematical conceptions. The use of CAS appears helpful to clarify problem solving strategies, but the adoption of higher order mathematical conceptions behind procedures seems to be limited. The impact of using CAS for students' mastery of algebraic equivalence has been explored by Ball, Pierce and Stacey (2003). The research pointed out that, in the context of solving equations, recognising equivalence, even in simple cases, is a significant obstacle for students.

One important and general issue related to calculators which is not considered in PME research is that stated by Kissane (2001):

> The development of the graphics calculator demands that we take a fresh look at the existing algebra curriculum, how it is taught and how it is learned, under an assumption of continuing and self-directed personal access to technology. Similarly, the development of the algebraic calculator suggests that we look closely at the content of our algebra curriculum and consider carefully a new role for symbolic manipulation, both by hand and by machine.

Concluding remarks on expressions and variables

From the review of research on teaching expressions and variables with technology, many issues and questions are to be faced or solved. Trends in emphasising students' learning and multiple views of concepts through multiple representations clearly appear, but so little, if not any, attention has been paid to curricular aspects and teachers' knowledge or teaching practice up to now. However, it is clear that the introduction of any kind of technology at school affects not only learning processes but even the conception and the control of the teaching situations, as outlined by Guin and Trouche (2000). For example, it requires efforts and time to be spent in the designing of suitable activities with technology and in instruction on the technology itself. This lack in research on school algebra was already stressed at the beginning of the '90s: "Unfortunately, there is a grave scarcity not only of models of the teaching of algebra but also of literature dealing with the beliefs and attitudes of algebra teachers" (Kieran, 1992). At that time the investigation was concerned with the learning and teaching of algebra without any specificity on the use of technology, but nowadays even considering the advent of technological tools the situation yet remains the same as then. As a matter of fact, some recent research also investigates curricular aspects, not in relation to an implementation of technology in the didactical practice. For example, Tsamir and Bazzini (2001) analysed the similar difficulties Israeli and Italian students had in solving standard and non-standard inequalities. The study was designed in order to extend the existing body of knowledge regarding students' ways of thinking and their difficulties when solving various types of algebraic inequalities. It was a result of the fact that in both Italy and Israel, algebraic inequalities receive relatively little attention and are usually discussed only with mathematics majors in the upper grades of secondary school. An open hypothesis is then that studies concerning curricular aspects do not generally pay attention to the use of technology.

From the viewpoint of educational research, an overarching question is how can we direct our use of the computer in mathematics education to the algebra of the future in addition to the algebra of the past and present (Tall, 1987a). Some related research questions we can raise here are the following:

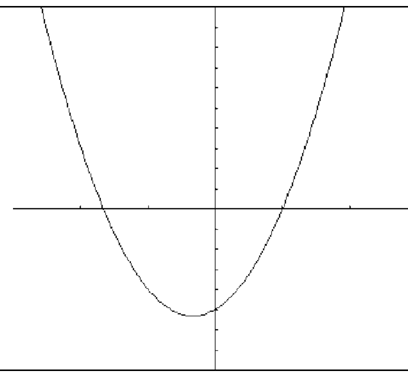
– In which ways does the use of technology tend to re-define the school subject of expressions and variables?
– Is the curricular role of elementary algebra changing as a result of the availability of new technologies?
– How could research of expressions and variables with technology inform the design of technology?
– What are the most urgent areas in research of expressions and variables with technology that can support teaching of school algebra?

TECHNOLOGY IN THE TEACHING AND LEARNING OF FUNCTIONS

In the conventional curriculum, early algebraic work tends to focus on the solution of simple equations in which a single unknown value is given a symbolic representation. This strand of work develops through increasingly complex situations, where nevertheless the aim is to find the value of one or more unknowns (for example in quadratic and simultaneous equations).

However, a parallel strand emerges from the early algebraic work in which the symbolization encapsulates not a single value but a variable or parameter, which represents a set of values in a domain or co-domain. This strand leads to the study of functions and graphs and is seen by many as the forerunner to the study of calculus. In this section, we review PME research on technology and functions.

We can appreciate functions through three dynamic representational systems:

(i) symbolic	$y = 3x^2 + 2x - 5$ *or* $x \to 3x^2 + 2x - 5$
(ii) graphical	
(iii) tabular	x y 0 -5 1 0 2 11 3 28

Much of the learning effort seems to involve assimilating these three types of representation into a meaningful and coherent whole. There is of course a fourth *verbal* representation, more often associated with older technologies, such as paper and pencil, than digital technology, which is our focus here.

We can now look back over the research in PME over the last three decades and identify three different approaches to the task of understanding how technology impacts upon this process of assimilation. One strategy has been to investigate the use of technology as a means of placing emphasis on one type of representation, perhaps simplifying others. The initial emphasis on one representation may be seen as a means of giving access to the notion of function before other types of representation are introduced. A second approach is diametrically opposite in that it has attempted to use the technology to support simultaneous connections between the various systems of representation. Finally a third approach has sought to consider how the learner uses the tool itself sometimes taking into account the broader setting. We will consider each of these approaches in turn.

Technology as simplifier

One clear affordance of technology is that it can reduce the demands of handling difficult numbers in a task. One such demand challenges confidence in calculations. Technology can be used to carry out or check calculations or graphs of functions. Mesa and Gómez (1996) presented a study based on the methodology of two groups of students: One experimental and one control. This study focused on students' responses to problems given in the classroom using graphic calculators. The authors analysed the different strategies applied by the students in the solutions and classified them on the basis of students' understanding and findings. It seems that technology was used more to verify strategies of calculations than to make conjectures and test them.

An analogous outcome was presented by Moreira (2002), who described the activities of a small group of students in a curriculum project, aimed at developing "democratic competence" in the context of mathematics education. The students were in the first year of a degree course in Management. They used Excel to work on problems about functions (graphs and calculations). The use of this technology was mainly oriented in two directions: The creation of a clear and well-presented graph of the function and the avoidance of fastidious calculations. The students verified claims about the size of the whale population involved in the problem. The potential of the computer was shown in the work of these students, who were able to verify claims about the whale population problem.

The above research shows how verification of calculations and graphs can be used in the graphing and tabular representation of functions, though it is not difficult to imagine symbolic manipulators being used to verify functions in the third type of representation. However, apart from simplifying the problem by easing the verification process, we are also able to simplify through the elimination of the need for symbols. How though do we input functions without the use of symbols or a symbolic language? We found in the research two main methods; one approach has been to use a device through which data is captured directly and transferred automatically to the computer. The other approach is to allow data entry by the student when that data has been generated during experimentation. In both cases, the aim is to study variation as a precursor perhaps to formal work on functions or statistics.

Data capture through the use of devices

Recent work includes an innovative type of experimentation with situations and representations: Using Microcomputer-based Laboratories (MBLs) to allow students to represent situations with devices that gather data in real time and present the data graphically. These technologies eliminate algebraic symbols as the sole channel into mathematical representation and motivate students to experiment with the situation—to analyse and reflect upon it—even when the situation is too complicated for them to approach it symbolically. Compared to working with visual representations of algebraic or numeric symbols, the visual analysis that is enabled by working with MBL tools is quite distinctive.

For example, Yerushalmy, Shternberg and Gilead (1999) used the mouse to draw trajectories of a "body" in motion so that the activities' focus is on the transition from the drawing action to an analysis of graphs of the mathematical functions. Other technologies that support links between body activity and "official" representations were studied by Nemirovsky, Kaput & Rochelle (1998) and Borba & Scheffer (2003). They suggested that it is possible to deepen students' connection with everyday experience through environments, which combine phenomena and their modelling. The interplay between notations, simulations and physical phenomena can be expanded by incorporating kinaesthetic activity, by empowering notations as controls over the creation of phenomena, and by comparing the physical with the virtual. In this sense we should consider the introduction of another type of representation, the kinaesthetic sense of function. The research of Nemirovsky and others proposes that the kinaesthetic representation might be a particularly facilitating starting point.

Nemirovsky (1995) reported a study of 10-year-old students who created "motion trips" on a meter tape by walking, running and stopping. They were then asked to represent these trips using tables and graphs. They were also able to use a program to invent such trips and then view them in various representational forms. More recently, 14 year-old students used a motion sensor in a similar way (Robutti & Ferrara, 2002) and were then set an interpretation task of a space-time graph. In this latter study however, the performance of the students was compared to that of a control group on the same task. The researchers concluded that the technology facilitated transitions between static and dynamic interpretations of the space-time graphs, leading to normalised meanings for the graph.

Two studies of the second type suggest that this facilitating effect of the technology is also apparent when the independent variable is not time. In one study (Nemirovsky, 1994), a high school student used a so-called "contour analyser" to trace a surface along a certain plane. The device generated a computer-based graph of height or slope versus position. The student learned about slope by striving to grasp what the tool does. This sense-making activity drew on previous knowledge, which had to be re-assessed. Noble and Nemirovsky (1995) used a motion detector as above but focussed on non-temporal graphs of velocity v position. The report describes the evolution of a high school student's thinking through his articulations in his attempts to match the graph with the motion of his car.

Kaput and Hegedus (2002) provided examples of classroom connectivity in which students discussed representations of families of functions with TI-83 calculators. They examined how the technological connectivity generated a personal identity as a resource for focusing attention and generating engagement with the tasks. A firm connection between this work and formal algebra was reported by Hegedus and Kaput (2003). Using a pre-test/post-test approach, they reported gains in learning through SimCalc, software which enables initial access to functions through virtual simulations of everyday situations, typically of time dependent phenomena. Their central claim was that combining the dynamic of SimCalc environment with classroom connectivity made possible significant improvements in students' performance on 10th grade MCAS algebra-related questions in a short period of time. The use of kinaesthetic approaches leads to a modelling perspective on functions in the sense that functions are seen as a means of exploring or analysing real world or simulated behaviour.

Data capture through experiments

One strand of research (Ainley, 1994; Pratt, 1994; Pratt, 1995; Ainley, Nardi & Pratt, 1998; Ainley, 2000) involved children carrying out an experiment that generated bivariate data and iteratively using scatter graphs to make sense of the emerging data. The kinaesthetic involvement of the children in the experiment supported their emergent appreciation of the analytic use of the graphs. In some cases, these experiments could be algebraically modelled. In such cases, the children were challenged to teach the formula to the spreadsheet. The approach then was to begin with kinaesthetic, tabular and graphical types of representation in order to open up the possibility of a symbolic representation. These papers report on how this approach allows the children to draw on a range of intuitions for interpreting scatter graphs, based on an intimate knowledge of their own experiment and spatial skills presumably drawn from other mathematical or everyday experiences. The students began to correct irregularities in the graphs (*normalising*) and construct meanings for trend. Connections between this and Nemirovsky's work have been made (Nemirovsky, 1998) indicating the value in paying close attention to students' ideas in the contexts within which they arose.

How are we to interpret the success of these technology-based studies in comparison to a range of prior studies that have shown graphing, and in particular the interpretation of graphs, as immensely challenging for young students? First it is worth noting that all of these studies involve experimentation. Whether the student is controlling the independent variable directly or indirectly, they do have a sense of active engagement with the technology. In some cases, the student is led towards trying to make sense of the technology itself. In other cases, it provides a tool to allow the student to pursue a more purposeful agenda by focussing on interpretation rather than on the technical skills of drawing a graph. Indeed, drawing on their work on graphing, Ainley and Pratt (2002) have discussed two constructs, *Purpose* and *Utility*, as providing a way of thinking about the design of pedagogic tasks.

The experimentation appears then to be allied to a greater sense of involvement in the sense of motivation, but also in the sense of bodily participation. In a PME Research Forum, Nemirovsky (2003) conjectured that perceptuo-motor activity (PCM) might be the root of mathematical abstraction, and that thinking might be PCM distributed across different areas of perception and motor action as shaped by early experiences with the subject. As evidence towards the substantiation of these conjectures, Borba and Scheffer (2003) reported on students using sensors attached to mini-cars to argue that technologies of information can create links between body activity and official representations. As further evidence, Rasmussen and Nemirovsky (2003) reported on students using a water wheel connected to real-time graphing software to conclude that knowing acceleration through a tool like the water wheel is something that grows and emerges in students across a range of representational systems.

Although there is a need for further research in this area, there seems reason to believe that, perhaps not only in the domain of graphing, we might design effective tasks that connect purpose to significant elements of mathematics by drawing upon perceptuo-motor activity.

This "function as simplifier" strand of PME research appears to have made considerable progress from a narrow perspective in which technology was essentially used to verify procedures to a proposal that we approach the teaching of functions through a further type of representation, that of the kinaesthetic, by using the technology to model behaviour in which we are personally and directly engaged.

Technology as integrator

Functions are often introduced in school textbooks through a static definition (static in the sense that this approach emerges from pencil and paper technology in which the definition can not be seen directly as having dynamic computational potential). Confrey and Smith (1992), illustrating their argument through the work of one imaginary student, claimed that it was important to build an understanding of functions through multiple representations and contextual problems before emphasis was placed on static definitions. An affordance of technology is to offer access to the various types of representation of function. This affordance has been widely exploited in PME research over the last three decades.

Excited by the possibilities inherent in the notion of multiple representations, researchers have developed distinctions around the notion of function. For example, some researchers have focused on the classes of translations between representational systems. Other researchers have focussed on the new ways that technology allows students to manipulate functions –mainly as graphical objects.

Translations between representational systems

Schwarz and Bruckheimer (1988) considered the transfer of knowledge between representations in a computerised TRM (Triple Representational Model: Algebraic, graphical and tabular) environment. Grade 9 children in junior high school were asked to use TRM to search for solutions. One group began by searching through the graphing representation before using the algebraic type of representation. Another group was asked to approach the problem in the reverse order. The authors concluded that they could sketch three cognitive levels of functional thinking: The numerical level where searching for a solution was systematic, the functional reasoning level where the search was systematic but logical sequences of computations were not used, and the dynamic functional reasoning level where the richness of the concept of function was understood and searching was efficient. They also concluded that focusing on graphing before algebra led to a higher level of functional reasoning as accuracy and convergence procedures transferred from graphing to algebra but not vice versa.

The research of Schwarz and Bruckenheimer leant towards a prescription favouring an introduction to graphing representations prior to algebraic (and note that in the previous section, there was some evidence to suggest the introduction of kinaesthetic aspects even earlier than this). However, as the power of technology has increased, software design has moved to a point where there can be increased fluidity between representations. Indeed, when representations are "hot-wired", it is impossible to detect any delay when one representation synchronises to reflect a change in another. A series of studies have found benefits in exploiting the connectivity between representational systems.

Resnick, Schwartz and Hershkowitz (1994) discussed how 9th graders, using graphic calculators to solve problems relating perimeter and area of a rectangle, found different ways to articulate arguments by shifting from one representation to another and noticing local and global properties. The authors proposed that there is a radical difference between paper and pencil representations, which are passive descriptions, and computer-based representations that are driven by actions. Schwarz and Hershkowitz (1996) reported a comparative study to probe into these differences. The control group used only linear functions and resisted using others, even when they were mentioned. The experimental group used a technology-intensive curriculum and developed a much richer variety of prototype functions to support reasoning and to exemplify strategies and properties. In 1997, Hershkowitz and Schwarz presented two studies on learning the function concept. In the first study, which was based on a questionnaire, the authors characterized the concept image of 9th graders after they had learned in an interactive environment, based on multi-representational tools and open ended activities. They found that these students, (a) used many examples, (b) provided rich justifications to answers, (c) showed flexibility within and among representations, (d) considered the acceptability of answers in light of the context, and (e) integrated prototypic examples with other examples. The second study was classroom-based with the same children. The focus now was on interactions between individuals, the problem situation, tools, and the community. Although the role of the computer might be seen as minor in this study. In fact the researchers claim that the internalised representations of manipulations aided the construction of hypotheses about the behaviour of the functions. Tabach and Hershkowitz (2002) traced how their students attempted to construct generalizations of growth patterns and used them to represent phenomena numerically and graphically, all through the use of spreadsheets. The report claimed that students constructed new knowledge and subsequently consolidated it. Smart (1995) reported a study in which girls had open access to graphic calculators during their mathematics lessons. The girls started to develop a robust visual image of many algebraic functions. Sutherland and Rojano (1996) looked at the impact of a modelling approach to understanding science concepts using spreadsheets. Students of two courses, one in Mexico and another in the UK, submitted pre-evaluation and post interview data. The authors claimed that spreadsheets helped students relate graphical and numerical representations, and make sense of the algebraic models of the physical phenomena. We have already discussed other aspects of the modelling perspective of functions.

Other studies though have alerted educationalists to some problems in using technology. Goldenberg (1997) illustrated a range of illusions inherent in the way that graphs could be presented. Perceptual strategies that were sufficient for interpreting scale and relative position in real-world spaces were inappropriate when dealing with the infinite and relatively featureless objects in coordinate graphs. In fact the prevalence of illusions appeared to be an issue for the reading of graphs in conventional settings as well as computer-based ones. However, as is so often the case, it was the use of technology that triggered awareness of the problem and arguably illusions may have been accentuated by the technology. Cavanagh and Mitchelmore (2000) investigated how students interpreted linear and quadratic graphs on a graphics calculator screen. They identified three common misconceptions: A tendency to (a) Accept the graphic image uncritically, without attempting to relate it to other symbolic or numerical information, (b) A poor understanding of the concept of scale, and (c) An inadequate grasp of accuracy and approximation.

Actions on graphs

When using conventional pencil and paper and technology, graphs are essentially an output resulting from the calculation and plotting of points. In that sense, they are an end point, offering no further actions that can be carried out on the graphs, without starting a new and lengthy procedure. Manipulations in algebra are reserved for symbolic manipulations, and the graphs are driven by them. Technology allows students and teachers to directly manipulate graphs of functions.

Confrey (1994) examined six different approaches to transformation (translations and dilations) of functions. She based her analysis on several years of experience using *Function Probe* (FP) in a variety of settings. Confrey constructed a schedule of transformations as defined by a set of parameters, for example A, B, C, and D. The paper presented this schedule as a framework for thinking about the teaching of transformation of functions and how the students might approach these problems. She claims that these six approaches are distinctive, each offering its own advantages and disadvantages. Borba (1994, 1995) studied the roles of visualization and direct actions on graphs in the FP environment. Borba's findings cover a wide range of aspects of multiple representation software, and in particular the study of manipulations of graphs. Borba's (1994) analysis was based on an episode with one student who predicted how the operation of "stretching" would change the graph. This episode served as an illustration of the different elements of the model. In the 1995 article, Borba described how a student, using multi-representational software, resolved a discrepancy in results by mentally adding features to the software design. Indeed, Borba refined a model for understanding in multi-representational environments and argued that students' reasoning can inspire software design.

Technology as instrument or mediator

Some research has looked broadly at the value added of using a piece of technology compared to conventional approaches. Thus, Guttenberger (1992) compared the performance of 31 Grade 11 students with access to a computer-based function plotter to those without. The post-test showed that these students performed at a higher level. Comparing students who used graphic calculators in activities on functions with others who did not, Gómez and Fernández (1997) found no differences between the two groups at the adaptation phase but significant differences were found at the consolidation phase. However, such research is limited in its ability to explain, either from the design or cognitive perspectives, how or why such improvements happened. For example, Dagher and Artigue (1993) studied 33 students, aged 16-18 years, working on second degree polynomials, and 21 students, aged 14 to 15 years, working on linear functions. These students were given a game, consisting of a curve that had to be represented algebraically. Points were awarded according to how little assistance was needed. Post test success on paper and pencil test about functions was much higher than on the pre-test. The authors noticed sudden crystallisations of thinking after which a clear game strategy emerged. By looking closely at the learning process in relation to the structuring resources in the game, the researchers were able to identify certain catalysts for the crystallisations. They noticed the importance of (i) meeting a particular parabola, (ii) changing one representation to a different form and (iii) locating specific points. This detailed analysis promotes a focus on aspects of the design of the game as well as how the resource mediated learning, in this case in the form of the breakthrough insights.

There has been little research on the design of tools to support the teaching and learning of functions. In an early theoretical paper, Lesh and Herre (1987) applied Dienes principles to the use of virtual instantiations of polynomials to propose that Dienes ideas, developed originally in the context of designing material manipulatives, have explanatory power even in the virtual setting. It is perhaps disappointing that this work has not been picked up by subsequent PME researchers in the field.

Rather more studies have focussed on mediating aspects. Lagrange (1999) reported on a study about the schemes of use introduced by students of the 11th grade, learning about functions with complex calculators (TI92). From the perspective of instrumental genesis (Verillon & Rabardel, 1995), the author's analysis of the development of the students' schemes of use in a pre-calculus course emphasized the role of calculators in terms of the mediation they offered to the learning process. He noticed, as others have done (see in § *Limits*), that students using symbolic manipulations saw the limit as an object but that they lost the sense of the process. He pointed out that the symbolic-graphic calculator could enhance the understanding of calculus concepts in terms of numerical and graphical representations before their symbolic form. The calculator acted as a mediator in the learning process and in this mediation it is by no means neutral. Meeting new potentialities and constraints, the students have to elaborate schemes of use potentially rich in mathematical meanings, a process that requires support and encouragement from the teacher.

A few studies looked specifically at general aspects of the tool reflected throughout the learning processes. Gitirana (1998) reported on students who interacted with three microworlds, all focusing on the learning of functions. The students' explanations suggested different conceptualizations of function according to the pedagogical and technical aspects of the microworlds and according to their interactions during the activities, thus highlighting a sensitivity of learning to the structuring resources in the setting.

Hershkowitz and Kieran (2001) focused on two different ways in which students used tool-based representatives: A mechanistic and a meaningful way. A case study with three Canadian students who worked on growth patterns of rectangle areas was compared with a prior study with Israeli students. The Canadian students applied a recursive expression technique and generalized it with the help of the tool. The authors questioned whether this may have hindered them from mathematizing the problem and from thinking in terms of an exponential function. Friedlander and Stein (2001) studied Israeli students' choices among tools (e.g. spreadsheets, symbolic calculators, graphic calculators) to solve equations. The results showed that students not only employed a variety of solution methods to solving equations but, more importantly, made connections between the representations afforded by different tools.

Concluding remarks on functions

In summary, there appears to be some consensus that technology can be exploited to privilege certain types of representation over others, focussing attention in specific aspects of function. Of the three standard types of representation (graphing, the use of symbolic algebra and tabularisation), there appears to be some evidence that students can use intuitive knowledge of the graphing aspect of functions to make sense of symbolisation more easily than vice versa. However, there also seems to be some evidence to suggest that a kinaesthetic approach may offer access at an intuitive level that can be utilised by allowing that type of representation to drive experience of the three standard representational systems. There has been considerable research effort to exploit the potential of technology to offer multiply-linked representations and some results seem to suggest that the linking of representations on screen may support such mental connections being constructed. However, too little is understood about this process. Some approaches have allowed these connections to be hot-wired, running the risk that the child may simply not attend to the changes being made. Others have tried to put the child in situations where they have to make the connections in order to pursue some broader objective. We need to know more about typical ways in which children make these connections. What are the critical parameters in enabling such learning and hat is the range of learning outcomes in such circumstances? There has been little PME research on the design of the tools. Given the wide interest in the situatedness of learning, it seems odd that there is little research into the relationship between design and learning about functions. Instead we have a setoff interesting examples of how innovative tools have been used without sufficient analysis of how the design intentions played out in practice. We argue that research of learning with tools cannot ignore the design of those tools.

Although we have, for the purposes of this review, treated the solving of equations as a separate activity form that of working with functions, clearly the broader pedagogic aim is to help children connect these two domains. In fact, the technology-based work on functions promises to have pay-offs for the understanding of equations. Yerushalmy (1997) and Yerushalmy & Bohr (1995) reported two studies, which were part of a three-year longitudinal investigation based on VisualMath, a functions-based approach to algebra. The former article reported significant changes in students' thinking about symbols, equations, and problems in context. This study included an analysis of common models of problem solvers, who were mainly low attaining students. The enhanced performance and thinking processes of students, who had previously been unsuccessful with algebra in general and with solving contextual problems in particular, has recently emerged from studies on the integration of graphing technology and contextual problem solving. Because students of VisualMath learned to view any equation in a single variable as a comparison between two functions, the second article posed the question of what the mental image for equations in two variables would be for these students. We therefore look forward to research which links the themes of expression, variable, function and calculus to inform our understanding of the longer term development of thinking about functions and its relationship to the affordances in the setting.

TECHNOLOGY IN THE TEACHING AND LEARNING OF CALCULUS

Looking back historically, there are three root aspects of calculus as a discipline (Kaput, 1994): One concerns the computations of areas, volumes and tangent, and first of all it was practical, as in the work of Archimedes. The second was a blend of practical and theoretical interest, involving the study of physical quantities variation. It began in ancient Greece with the mathematization of change and had a golden period with Newton's study of motion, with the application of new mathematical tools for calculation. While the third was inherently theoretical, beginning with Zeno's ancient motion paradoxes and continuing with the complete formalization of analysis in the 19th century, and has been recently completed with the non-standard analysis. These three roots, as Kaput (1994) wrote, "interweave complexly, and the story of their interweaving is still being written".

The introduction of technology in calculus in recent years touches these three roots, exploring the potentialities of them with different supporting contexts, as for example: The numerical calculation of areas, volumes and tangents (with programming languages, spreadsheet or software); the elaboration of data taken from measures of quantities (by sensors connected on-line with calculators), for investigating the relations and the variations of these quantities, and the approach to continuous and to infinitesimal analysis, through functions and variables managed in a symbolic way (CAS).

We develop at least three elements of analysis: Epistemological, cognitive and curricular. In calculus, the power of technology is particularly important to facilitate students' work with numerous epistemological discontinuities such as discrete/continuum, finite/infinite, determinate/indeterminate, and so on which also relate to other subjects (e.g. arithmetic, algebra, analytical geometry) previously learned by students. These discontinuities can remain epistemological obstacles not understood by the students or may be overcome on the road towards the construction of concepts. "The construction of pedagogical strategies for teaching students must then take such obstacles into account. It is not a question of avoiding them but, on the contrary, to lead the student to meet them and to overcome them" (Cornu, 1991, p. 162). The obstacles create the cognitive demand for an overcoming, and only with deep research in mathematics education we can understand them and try to find some solution. The same solutions can then be used to plan curricular projects: In this way, research can be useful to teaching practice.

Our analysis focuses on the mediation of technology and the role of the teacher about these epistemological obstacles: Limits, derivatives, infinite sums, integrals.

The approaches to calculus in the past three decades, according to the parallel introduction of technology at school, have been different: First of all, numerical investigations could be possible, due to the programming languages used to solve problems relative to functions in a numerical way. Then graphical environments rendered possible the investigations on the shape of functions, both at a global and at a local level. Finally, symbolic performances became possible, for the introduction of Computer Algebra Systems in computers and calculators. These environments were used in an independent way from each other, and also simultaneously. But the big revolution in teaching mathematics with technologies was the introduction of dynamicity in software: A dynamic way to control and master the virtual objects on the computer let the student explore many situations and notice what changes and what does not. And the mathematics of change is the first step on the road to calculus. We can intend change at a numerical level, as well as the graphical or symbolic level. If some researchers concentrated their attention on one of these levels, others were more interested in the simultaneous use of them in order to integrate their potentialities.

In using technology at school there was the necessity to adapt the mathematical activities to the potentialities offered by the software in use, and sometimes the limitations of this software could influence the construction of a concept (as mentioned by Magidson, 1992) referring to slope, conceptualised by the students as a value not more than 8. This necessity, over the years, has been decreasing and the idea of a generic organiser entered the scene (see § *A dynamic view of Algebra*). It embodies a theoretical structure and the user may come to understand it using the generic organiser in specific examples. The existence of such a structure absolutely does not guarantee that the user will abstract the general concept. So technology can never guarantee learning or take the place of the teacher. Surely it can help students to overcome certain difficulties, but research shows that it will be effective only within a coherent teaching-learning context. Better if this is done in a social way, working in small groups and then sharing the ideas in a class discussion, guided by the teacher, as reported in many articles.

Limits

Traditionally, limits are often introduced through the epsilon-delta definition, which compels the students to reason without the consideration of functions as maps between x and y values in the Cartesian plane, and, on the contrary, fixes their attention on intervals on the y axes, and finding a corresponding interval on the x axes. This way of reasoning is different from previous algebraic thinking, and breaks a balance about functions, often causing cognitive difficulties, as witnessed in literature. For students it is very difficult to reverse the order (Kidron, Zehavi & Openhaim, 2001). There can be considered three paradigms for teaching limits: A (formula-bound) dynamic limit paradigm, a functional/numeric computer paradigm and the formal epsilon-delta paradigm (Li & Tall, 1993). The first is based on arithmetic and geometric progressions and convergence of the latter. This approach emphasises the potential infinity of a process that cannot be completed in a finite time. The second is based on programs written in Basic that give as output a certain value of a sequence. They support a numerical exploration of sequences, and ground limit as a procept in the sense of Cauchy, being the terms indistinguishable at a certain step, instead of evaluating the limit value. The third is the traditional approach, based on formal definition. The authors use the second paradigm for experimentation and finally recognise that difficulties in passing to the formal definition remain for the epistemological discontinuities in the concept definition itself.

As in the previous case, where alternatives routes to traditional teaching are used in calculus with the integration of technology, many other studies have been carried out in this direction. They outline that technology by itself does not promote change (Valero & Gómez, 1996). What promotes change is the curricular project in which technology is inserted, and in particular, the didactic sequences planned by the teachers in order to introduce calculus concepts, which use technology as a support. Planning these sequences involves re-considering activities, methodologies, lessons, learning contexts and all aspects of teaching, integrating the “old” and the “new”. So, it means that we have to make important choices about technology: How, when, why, and what kind of technology we must use.

According to the kind of technology used, and, more importantly, the type of use of it, a teacher may convey a given meaning in a particular way. For example, as said before, it is possible to teach the concept of limit centring on the idea of “getting closer”, or on the “limit value” as the object that constitutes the limit. Students are usually familiar with the former, which is more intuitive than the latter. Using Derive to calculate limits, it is possible to orientate students with the latter meaning, because they are able, with this software, to see the numerical value of the limit. On the other hand, this approach leads students to consider limit as a static value, losing the sense of the approaching process. They acquire the advantage of seeing the limit as an object, but they miss the sense of the process that lies beneath the object itself (Monaghan, Sun & Tall, 1994). In other approaches, the conceptualisation of limits may pass through a discrete process (e.g. using graphic calculators), that here have the role of reinforcing former limit conceptions based on the process of "getting closer" to something. Conversely this line of teaching seems to make more difficult the construction of the meaning of limit as a concept of "limit value", as it follows from the formal definition (Trouche & Guin, 1996).

Between these two ways of conceiving the limit (namely, process of getting closer to something or limit value), there seems to be a gap, which is difficult to bridge. It corresponds to an epistemological discontinuity, which is also present in the history of calculus, in which the static epsilon-delta definition of the object limit was only the last step of this history, which started with the intuitive process of getting closer. calculus is difficult for our students, because it reflects the great difficulties faced in the history of mathematics.

What is the role of technology in these different approaches to limits? There have been cases put forward for using Derive to calculate limits (Monaghan et al., 1994) and exploring functions by conjecturing, solving and checking (Trouche & Guin, 1996).

A possible choice for teachers at secondary school level, where calculus is only partially taught is to support the conceptualisation of limits with intuitive basic ideas about infinitesimals, together with a brief history of them and without a strong and abstract formalisation. Infinitesimal conceptions have been used to teach limits (Cornu, 1991). Adding to the mathematical legitimacy of infinitesimals, recent research (Milani & Baldino, 2002) offers a challenge to mathematics education: "What if, instead of waiting for the infinitesimal conceptions to emerge, we stimulated them? What if the students became aware of the abyss between mathematics and its applications produced by the, now unfounded, discrimination of infinitesimals? Will such an awareness stimulate the transposition of the obstacle towards the understanding of limits or will it create new obstacles?" (Milani & Baldino, 2002, p. 346). With the technological support of CorelDraw zoom, four freshmen in a calculus course for physics students first were introduced to the basic ideas of infinitesimals and their use in calculus, then they demonstrated such ideas to the whole class. For example, the zoom function of CorelDraw has been used to visualise the merging together of a curve and its tangent line at the point P, showing that curve and tangent appear as parallel straight lines at $P+dP$, and so on. Not only was an instrument introduced in order to use infinitesimals, but also a sign: ≈, to indicate "infinitely close to". These are the words of a student after this experience: "In physics we have to imagine the situation, we have to imagine what happens in a very small space, with tiny dimensions. The zoom helps. In the physics class the teacher also spoke about the zoom, taking an infinitely close view, but it was not clear for all the students." This research shows the mingling of mathematical, continuous, infinitesimal conceptions with the physical, discrete, subatomic reality. The authors come to the conclusion that "the whole research process led the students to spontaneously enlarge their concept images so as to incorporate elements from the microscopic physical world among infinitesimals." (Milani & Baldino, 2002, p. 346).

At a higher level of content, a remarkable example is offered by Kidron et al. (2001), who studied the possibility to enhance students' ability for passing from visual interpretation of the limit concept to formal reasoning; they used symbolic computation and graphics. The task consists in approximating functions by Taylor polynomials. The students used two methods to approximate a given function by polynomials: Analytical and algebraic. In the analytical method they approach the notion of *order of contact* with Mathematica. In the algebraic one they *follow the original text of Euler* (1988), applying his approach to represent infinite sums with Mathematica. Both methods lead to the coefficients of the Taylor series but with different features: The analytical one describes the *process* of the different polynomials approaching a given function; the other represents the polynomials with "an infinite number of terms" as an *object*.

The Mathematica software was useful both in the calculations of the algebraic approach and in the analytical method; for example, a sequence of plots helped illustrate the fact that, in a given interval, the higher the degree of the approximating polynomials the closer the function sin(x) and the polynomial are. This animation generated by the software helps students in the conceptualisation of limit because it permits them to see the dynamic process in one picture.

The static object introduced by Euler becomes a dynamic process of convergence to a function: In these activities, the students used Mathematica to visualise functions and to construct animations, but this was not enough in order to transform a process into the concept of limit. So the students "had to *interact* with the dynamic graphics, to *have control over the dynamic representations*" (Kidron et al., 2001). In this article the process of instrumentation (Verillon & Rabardel, 1995) is described, by the schemes of use activated by the students, who transformed the artefact Mathematica into an instrument for conceptualisation.

One year later one of the authors (Kidron, 2002) continues the investigation of this problem, adding a new element to her previous theoretical framework: The embodied cognition analysis of infinity (Lakoff & Núñez, 2000). The analytical approach is seen as a way to represent *potentially infinite* processes, while the algebraic one as a way to represent *actual infinite* sums as objects in the author's mind: "The illusion that the infinite calculations are performed at once might facilitate the transition from symbolic manipulation to a symbolic object" (Kidron, 2002). The researchers' aim is to examine to what extent the interrelationship between the two approaches and the different uses of the software helped the students to perceive an infinite sum as the limit of an infinite process. It seems that using different approaches the students were able to understand that an infinite sum is not necessarily an expression that tends to infinite, that it could be equal to a given number; that it can be conceived as an object; and that it might be rendered easier if we apply an algebraic approach to it, as Euler said. Particularly, a sort of balance between the conception of an infinite sum as a process and as an object seems to be supported by the use of the software, together with the didactical methodology applied.

With regards to the other themes related to the limit of functions, namely the problem of continuity, we do not yet have studies, apart from Vinner (1987) on students' competencies about continuous and discontinuous functions, investigated through a questionnaire. It turns out that, although they succeed in the identification task in the common cases, they fail very often to justify their answers. Moreover, their use of the limit concept is quite fuzzy and they often rely on irrelevant argument. As a result, the level of their mathematical reasoning is quite inadequate.

Derivative and integral

Teaching derivatives and integrals raises issues analogous to the ones on teaching limits: The intuitive approach versus the formal one. It can be seen as a part of the theme of functions, as it is often so presented in the literature. The studies of Tall (1989) grounded these contents on the cognitive roots of local straightness of a function and area under a graph, for derivative and integral respectively. Ensuing literature followed these studies. The use of new technologies demonstrates the possibility of grounding the concepts of derivative and integral on simpler ones, implemented through various environments such as the numerical one, the graphical one, and, last but not least, the symbolic one. The suggestion for mathematics education research is a progressive conceptualisation toward the abstract definition and use of these concepts, passing through activities of exploring, conjecturing and testing conjectures. The use of software such as Graphic Calculus can be done at different levels, as Tall shows in his articles. In general, it can be used in a rich way employing the dynamic images in a fruitful cognitive way. For example, a function and its gradient function can be plotted together on the same screen, and moving a chord on the first renders possible the generation of the second. As one student put it: "I never understood what it meant to say that the derivative of sin*x* was cos*x* until I saw it grow on the computer". The verb "grow" really embodies the dynamic nature of the gradient function, as seen on the screen. As students explore and enrich their concept images in a more personal way, they seem to regard the computer as an authority that does not present the same threat as the teacher. They seem far more willing to discuss conceptual difficulties thrown up by the computer than they would difficulties in understanding a teacher's explanation (Tall & Sheath, 1983).

Another study at secondary school level over several years shows that Graphic Calculus can be used as a generic organiser in different ways. In order to introduce derivative, it can be used to magnify graphs, allowing the student to see what happens to functions when their graphs are magnified. Many graphs look "less curved" under higher magnification. After using this organiser (with the guidance of the teacher in order to explore different functions, with or without the property to become straight when magnified), the student can look along a graph, "magnifying it in his mind's eye, and seeing the gradient vary" (Tall, 1985). The use of the same software as another organiser is to observe the variation of the gradient of a graph, moving the chord through two near points along the curve, changing the x of the first point but not the proximity of the two. In this way, the notion of gradient can be constructed in a dynamic and a global way, even if the final construction, the curve of the gradient, is a static picture on the screen. Thus the concept image of gradient can be created both as a process and as a concept, and therefore as a procept.

Linked to the derivative and the slope of a function, we have the concept of tangent to a graph at a certain point. As pointed out by Artigue (1991) there are many ways to conceive the tangent to a curve. For example: As a line passing through a point but not crossing the curve in a neighbourhood of it, as a line having a double intersection with the curve at a certain point, as a line passing through two points infinitely close to a point on the curve, and so on. If these different points of view can coexist in the mind of a mathematician, it may not be the same for a student, because their contemporary presence can be a source for misconceptions, obstacles and conflicts. Mathematicians analyse a concept in a formal manner, producing a hierarchical development and linking different concept definitions in a proper way. But this way may be inappropriate for the developing learner. The computer can help to overcome such difficulties, by giving the possibility of introducing new concepts that had previously seemed extremely abstract to pupils, or by offering them the opportunity to coordinate different perspectives on the same concept. A study carried out in three experimental classes (Tall, 1987b), aimed at the construction of the concept image of a tangent made use of the computer to draw a line through two very close points on the graph of a function as part of a broader introduction to the idea of gradient. The students experienced the presence of a tangent in some situation and the absence of it in others, such as the case of the absolute value of sin*x*, which has "corners". The study emphasised the difficulties embodied in the tangent concept, but suggests that the experiences of the experimental groups helped them to develop a more coherent concept image, with an enhanced ability to transfer this knowledge to a new context.

The use of graphic calculators opens up the possibility to represent functions and to pass gradually from functions to their derivatives. A research report on first year Biology students focused on functions and derivatives with graphic calculators, Borba and Villareal (1998) point out that the calculators were always present conceptually, if not physically, and did not bar other media from being used. The evidence suggests that the technology does not merely supplement but actually acts as a reorganiser of cognition.

Exploiting multiple representations offered by TI92 calculators, Kendal and Stacey (2000) report on an introductory program of differential calculus at grade 11. The aim of the teaching sequence was the introduction of competencies on derivative and proficiency with various representations. The paper reports on the conceptual understanding of differentiation by two different classes of students. The competencies were tested by a set of items designed to measure levels of understanding with respect to the numerical, graphical and symbolic environments of the calculator. To this aim, different types of questions have been developed, as for example: "Find a rate of change" in the numerical case, "Find a gradient" in the graphical case, and "Find the derivative" in the symbolic case. They were based on the fact that: A numerical representation of derivative (derivative calculated at a point) is approximated by a difference quotient; a graphical representation is given by the gradient of the tangent to the curve at a point; a symbolic derivative is determined as a function manipulating formulas or as the limit of a function.

In a study presented by Ubuz and Kirkpinar (2000), first year undergraduate students were engaged in learning the concepts of calculus, particularly derivative. The authors analyse the factors contributing to learning calculus in a computer-based learning environment (Interactive Set Language ISETL and Derive). The outcome shows that there was a significant improvement in learning derivative in general in the graphical interpretation and in the use of the definition.

Integral are usually the last element in the curricular sequence of calculus, after limit and derivative, but with the use of technology a didactic inversion may offer an appropriate and attractive cognitive approach. In this case, as in the case of derivative, research demonstrates that introducing limits in a formal way only at the end of calculus, after derivative and integral, can be the better choice from a cognitive point of view (Tall, 1986a). The differentiation-integration concept takes place first at the numerical and graphical level, exploiting the computer, starting from physical or mathematical situations, than can be managed at the symbolic level. The area under a simple graph such as $f(x) = x^2$ may be approximated by dividing the interval into n equal width strips and adding together the "upper" and "lower" rectangular approximation in each strip. Knowing the formula to simplify the sums, it is possible to see what happens as n gets very large. In 1635 the Italian mathematician Cavalieri demonstrated his computational facility by performing the calculations for all powers of x up to 9. It was excellent and hard work, but today this can be done with the help of computers. With an algorithm in a programming language it is possible to see the results of these sums, but with this method the process of calculation of intermediate sums will be lost. Another way to approach the problem could be to use Graphic Calculus (the environment Area). In order to follow the process in each step: Drawing the graph of a function, calculating the area under a function. Students may explore many possibilities in a short time, choosing the function, the interval where to determine the area and the strip. Then some intriguing explorations can take place, as for example what about inverting the two extremes of the interval? The software dynamically shows a negative step and the picture of the area approximations builds up from right to left (Tall, 1986b). Another dynamic feature of the software permits observation of the generation of the integral function in real time, having fixed the interval of integration and the step. Another stage of the exploration can be the change of the number of steps, and one more could be the discovery of the addition property for integrals, starting from particular cases. This path tends to the fundamental theorem, as a final aim of a sequence of meaningful activities that can prepare students to the more formal approach to theory.

A paper by Hong and Thomas (1997) focuses on a significant improvement of conceptual thinking in integration through the use of computers and curricular modules of work in classroom. The use of these environments favoured an enhanced proceptual understanding with a tendency to understand in a concept-oriented manner, rather than as rote processes. In contrast, the control group of students with their traditional learning of calculus often experienced no change showing the same misconceptions in both pre- and post-test.

Implications for teaching and curriculum

The role of technology in supporting students engaged in mathematical activities may differ according to many variables: The students themselves and their background, the task, the mathematical context, the class context, the kind of technology used, the teacher's use of technology, etc. Given this complexity it is difficult to find patterns of use. For instance, students might sometimes use technology in a certain way because of certain teacher's beliefs. Let us review the role of the teacher in introducing technology in a classroom.

A case study in which three teachers were involved is described by Kendal and Stacey in order to find differences in the approach to calculus by students who had full access to calculators with CAS in the classroom, at home, and during tests (Kendal & Stacey, 1999). The study shows that each of the three classes obtains similar average scores of the test. However, they made very different use of the CAS environment and performed differently on the items of the test. The authors pointed their attention on "privileging", as the act that reflects the teacher's underlying beliefs about the nature of mathematics and how it should be taught. Privileging is derived from the interplay of teachers' beliefs and interrelated knowledge sources (i.e. content, content pedagogical, pedagogical); it is moderated by institutional knowledge about students and school constraints; it is shown through teachers' practice and attitudes and it is highly influential in students' learning. The study demonstrates how teachers' privileging can have an impact on students' learning and influence it. The great potential of CAS in providing multiple representations of mathematical concepts has been differently implemented by the three teachers. These differences have been translated into substantial differences in how their students solved problems and what they understood. For example, the students of class C understood what to do in algebraic contexts so they could compensate for poor algebraic skills by appropriate use of the calculator and by substituting algebraic with graphical procedures.

The same authors present a case study in which two teachers were involved (Kendal & Stacey, 2001): They discuss the classroom process data, referring to a 25 lesson course on introductory differential calculus to 16-17 year old students.. Analysis shows two distinct teaching styles, methods, representation preferences, functional and pedagogical uses of technology (the environment CAS in TI92 calculators). In their study, the authors concentrate on three components of privileging: Teaching approach; calculus content; use of technology (evidenced by the nature of use of the CAS calculator). Analysing the three aspects of each teacher's privileging during the teaching of an introductory calculus unit, the study monitors the changes that occurred over two years and explores the impact of new knowledge and a new situation on the changes in technology privileging, linking them to each teacher's beliefs and pedagogy. The results are that the two teachers changed technological privileging according to their prior beliefs and knowledge.

Other studies (Valero & Gómez, 1996) about teacher's beliefs in classrooms that used graphic calculators for functions, pointed out that there are real modifications in actions and expressed opinions on the role of technology in teaching mathematics but not in thoughts. The methodology for collecting data, based on interviews aimed at identifying the teacher's position on it, showed that some destabilisation in the belief system may occur, but that no significant change arises. The authors remarked the conclusion that technology itself does not promote change, neither in learning, nor in teaching.

The use of the same technology in a pre-calculus course is presented by Doerr and Zangor (1999), who analyse the role and beliefs of the teacher related to the patterns and modes of students' use of graphic calculators in supporting their mathematical activities. They found a set of ways in which the tool is used by the students and related mathematical norms. The teacher's confidence, flexibility of use, and her awareness of the limitations of technology itself, led to the establishment of: A norm that required results to be justified on mathematical grounds; multiple ways for visually checking hypothesised relationships between variables, a shifting role for the calculator from graphing to checking, and the use of non-calculator strategies for periodic transformations.

Pre-service mathematics teachers, most of who were concurrently engaged in their student teaching experience, were observed in order to study the impact of a multimedia teaching case on their professional development (Doerr, McClain & Bowers, 1997). The study presents the benefits and the limitations of the multimedia case.

Another important level of implications deals with curriculum design. In a period such as the last decade, when all over the world there have been new curricular projects in mathematics (see for example NCTM, 2000; Anichini, Arzarello, Ciarrapico & Robutti, 2004), it is important to reflect on the use of old and new symbols by students in order to support their construction of meaning. "How much and what kind of actions on formal symbol systems are needed to support the mental construction of objects that can in turn serve as referents for new symbols and systems of reasoning?" (Kaput, 1994). The passage from process conception to conceptual entity is difficult (see § *Limits*), and it is traversable with appropriate forms of deliberately designed experiences, as defining and manipulating wide varieties of function in a computer environment, possibly starting from perceptuo-motor activities, as shown in many cases (Arzarello & Robutti, 2003; Nemirovsky, 2003; Rasmussen & Nemirovsky, 2003). The problem of studying students' conceptualisation in terms of a semiotic approach has particularly pointed out by Radford (Radford, Demers, Guzmán & Cerulli, 2003; Radford, Cerulli, Demers & Guzmán, 2004) and by Arzarello (2005), and their studies converge on the kind of activities where students are involved and on the theoretical analysis, based on the objectification of knowledge, as making something apparent in a social way. These experimental and theoretical studies show their power in the didactical implication on the mathematics curriculum. As an example, the very recent curriculum proposal made by UMI in Italy (Anichini et al., 2004) realises the findings from these studies and points out the importance of the mathematics laboratory, intended as a methodology based on activities with materials and artefacts, where students are actively engaged in social work.

Concluding remarks on calculus

> "As for the precise contents envisaged by the research, even where the works concern different levels, one finds again common preoccupations:
>
> - Concern with developing a functional approach,
>
> - Concern to focus the notion of derivative on the existence of a good approximation of the first order, the computer allowing exact visualization of this property by magnification of the graph, even before the notion of limit is mastered." (Artigue, 1991)

Calculus needs to be studied across many years of school, from early grades onward, much as a subject like geometry should be studied. Hence its many purposes should be examined, not merely its refined methods. But most especially, its root problems should take precedence as the organizing force for curriculum design.

The power of new dynamic interactive technologies should be exploited in ways that reach beyond facilitating the use of traditional symbol systems (algebraic, numeric, and graphical), and especially in ways that allow controllable linkages between measurable events that are experienced as real by students and more formal mathematical representations of those events. (Kaput, 1994).

To educate students to observe numerical sequences, to see a graph and to read through symbols seems to identify the most common feature of the articles in this review.

FINAL REMARKS

The three main sections above have raised questions or drawn conclusions in relation to research on the use of technology in expressions and variables, functions and calculus. We do not therefore intend to summarise or repeat those conclusions here. In this short section, we constrain ourselves to a few overarching comments.

From the technology perspective

We can outline general trends relative to the implementation of technology for the teaching and learning of all the subject matters considered in this review. In fact, we have seen that new technology allows for dynamical approaches to the major concepts in algebra as well as in calculus, contrasting the static existence of traditional paper and pencil practices. Equally, the power of linking multiple representations, rich in terms of interactivity, is made explicit through the use of digital technology. As a consequence, interactivity and dynamicity are two features for which technology promises a wide potential, providing attention to the construction of meanings more than to the manipulative aspects.

We have also seen changes in the kinds of technologies used over time: Programming languages, such as Logo, Pascal and Basic, flourished toward the end of '80s as an approach to addressing specific contextual knowledge, such as that of variable, or function. But, at the end of the decade research interest in programming, at least as a medium for learning concepts, has all but vanished.

An open issue is then whether this trend is mostly a matter of fashion, or whether somehow the complexity of learning the particulars of the language was seen as counter-productive to the development of general cognitive and thinking skills through programming. Furthermore, software in which particular commands allow students to operate directly on mathematical objects and to see changes as results of their actions offer an immediate perception that can support the construction of those specific concepts, though, arguably, may be more constrained in terms of rich extensions.

From those years on, research increasingly focused on the development and the experimentation of new packages, games and tutorials, with the aim of improving the learning of mathematics, with specific didactical aims. Such programs are centred on the construction of meanings for circumscribed mathematical objects (for example, geometric figures, functions and their transformations, vectors, ...). But they can also be oriented to the construction of a 'piece' of theory, around the notion of theorem, for example, or proof, or deductive system. These programs are generally more 'mathematics-oriented', than open spaces where many different kinds of operations (related to more than one subject matter) are possible, such as a spreadsheet can be. We are referring in the first case, to L'Algebrista for example, and in the second case to CAS environments, which are paradigmatic cases of the introduction and diffusion of new technology over the last decade.

Implications for teaching practice and research

The review raises a number of issues, often with no answers at present, and indeed these issues often raise further questions. Indeed, Hershkowitz and Kieran (2001), set out the contextual factors of various origins that one should consider in the design as well as in the study of a classroom learning activity in a computerized mathematics learning environment: "(a) The mathematical content to be learned and its epistemological structure; (b) The learners, their mathematical knowledge culture, and the history with which they started the researched activity; (c) The classroom culture and norms, the role of the teacher, the learning organization –in small groups or individually–, etc.; and (d) The potential 'contribution' of the computerized tool" (ibid.). We have seen in this review that it is quite complex, even from the research point of view, to analyse an integration of all the variables, in order that they can be let fit together in a harmonic whole. In the practice of teaching, the implementation of technology "forces reconsideration of traditional questions about control and the social structure of classrooms and organizational structure of schools" (Kaput, 1992). The curricular choices and beliefs of teachers acquire great relevance in these terms. Substantially, technology re-awakens us to the complexity underlying the teaching and learning of mathematics, re-posing age-old questions that regard: "Educational goals, appropriate pedagogical strategies, and underlying beliefs about the nature of the subject matter, the nature of learners and learning, and the relation between knowledge and knower" (ibid.).

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