The Royal Society

JMC

Joint Mathematical Council
of the United Kingdom

Teaching and Learning Algebra pre–19

Report of a Royal Society/JMC Working Group

PREFACE

The Joint Mathematical Council in 1994/95 arranged two seminars to enable discussion about the apparent lack of articulation between mathematics taught at school and that required by higher education. The second seminar organised by Professor Margaret Brown, the Chairman of the JMC, and devoted to problems with algebra, took place in January 1995.

As a result of these seminars a working group was established under the auspices of the Joint Mathematical Council and The Royal Society to make recommendations about the teaching of algebra based on an examination of relevant evidence. The membership of the group was chosen to reflect the many different interests and phases in education, including schools, colleges and universities.

We wish to express our thanks to all members of the working party, and especially to the chairman, Professor Rosamund Sutherland, for their work in producing a most useful and carefully researched report with well-founded recommendations. We hope and expect that this report will be effective in improving the mathematical education of pupils in the United Kingdom.

We wish to record our appreciation for the support given by the London Mathematical Society in assisting with the dissemination of this report. We believe it deserves to be read by all with an interest in mathematical education.

The report has been approved for publication by the Councils of The Royal Society and the Joint Mathematical Council.

Professor J.C. Robson
Chairman, Joint Mathematical Council of the United Kingdom

Sir John Horlock, F.Eng, F.R.S.
Treasurer and Vice-President, The Royal Society.

"A student who knows only Arithmetic is quite right to say that 5 – 8 is impossible; but this impossibility is a gateway that leads to new knowledge, a stile that has to be got over, leading into a meadow in which fairer flowers grow than in the field of Arithmetic that is left behind, a passage through the looking-glass into a fairyland". **(Hudson, 1888, page 134)**

KEY CONCLUSIONS

The nature of school algebra

We identify three important components of school algebra: Generational activities—discovering algebraic expressions and equations. Transformational rule-based activities—manipulating and simplifying algebraic expressions, solving equations, equivalence and form. Global, meta-level activities—ideas of proof, mathematical structure and problem solving. Our overall conclusion is that, in England and Wales, an overemphasis has been placed on generational activities and that other aspects of algebra have received too little attention.

Algebra as a language

The algebraic language is required in order to develop awareness of mathematical objects and relationships. Without appropriate emphasis on the symbolic language such essential ideas as algebraic equivalence cannot be learned. It has to be accepted that pupils will make mistakes with the algebraic language; they must have extensive feedback on these mistakes before they reach post-16 education.

Changing emphasis in school algebra

Over the last 10–15 years a particular approach to algebra in schools has developed. It is characterized by an emphasis on problem-solving in real-world situations, an emphasis on relating algebra to pupils' informal methods and a de-emphasis of the role of symbols. For example, activities such as generating expressions from patterns, and the use of trial and improvement methods for solving quadratic equations have been emphasized. While these are valuable, they are not algebraic activities. We conclude that the National Curriculum is currently too unspecific and lacks substance in relation to algebra. The algebra component needs to be expanded and elucidated—indeed rethought.

Implications for national curricula

We recommend a reworking of algebra within the National Curriculum and a critical appraisal of the notion of levels on which the National Curriculum is based. The structuring of the National Curriculum should take into account both the need to preserve mathematical coherence and a consideration of how pupils learn algebra. We urge that the interrelationship between levels of attainment and Key Stage tests be re-examined. We suggest that more research is needed to understand the relationship between what algebra is taught and what is learned. In the A-level common core, content should not be separated from modes of algebraic activity. We also urge that there should be a coherent and clear algebra curriculum for the mathematical element of vocational courses.

Implications for the timing of algebra teaching

We have identified a range of activities which we consider to be precursors to algebra. These should take place both in primary and early secondary schools. We recommend that algebra is introduced from the beginning of secondary school, with more emphasis being placed on all aspects of algebra. At A-level, many students have to devote valuable time to the development of algebraic ideas at the start of their course. We recommend that post-16 institutions develop bridging algebra courses for some students before they start A-level.

Implications for teaching algebra

To be effective a teacher has to be aware of pupils' individual approaches as well as orchestrate learning so that pupils develop knowledge of mathematics that is recognized by communities outside school. Algebraic problems within school will always have to be contrived when relating to the real world. We believe problem situations have to be devised in which it makes sense to introduce algebraic concepts, and in which teachers are not fearful to talk about something which pupils cannot yet know about.

Implications for assessment

Current assessment practices in mathematics tend to place more emphasis on correct answers than on the process of solution. It is the latter which is crucial to algebra. The effects of form of assessment on learning mathematics needs to be investigated. We recommend that more attention be given to assessment design in order to promote algebraic activity.

Implications for the development of curriculum materials

Those involved in the presentation of mathematics to pupils need to reflect carefully on the likely learning effects of the presentation they choose. A mechanism has to be found that enables feedback on what pupils learn from these materials to be taken into account.

Implications regarding new technologies

In England and Wales many changes to the curriculum have centred around new technologies. Banning the calculator from a Key Stage test will not result in pupils changing their well-established method within the test. Banning pupils from ever using calculators in school in not sensible or practical.

Work with certain types of symbolic computer environments can support pupils to learn crucial algebraic ideas. We should try to capitalize on this possibility. More research is urgently needed on what pupils learn through using algebraic calculators.

Paper technology did not preclude teachers from asking their pupils to work mentally. Computer technology should not preclude teachers from asking pupils to work with paper. In England and Wales, current financial constraints often inhibit teachers from attending courses on the use of computers for teaching mathematics. This should be attended to.

Implications for teacher education

Teachers need support and guidance in order to recognize the essential nature of algebraic activity. We recommend that resources are made available to develop materials and courses to facilitate this, particularly through in-service training.

Implications for decision making

We urge that more reflection and analysis is built into the system. This requires time. It also implies the need for some body with an overall co-ordinating responsibility for mathematics from 5 to 19. We should not experiment 'on the job' with our future populations.

INDEX

1. Setting the Scene

2. The Pre-16 Curriculum

3. The 16–19 Curriculum: A-level Provision

4. Vocational Provision: 14–19

5. Conclusions

1. Setting the Scene

1.1 Introduction and background

This report of a working party set up in July 1995 is concerned with the teaching and learning of algebra. The aim of the working party was:

> *To make recommendations about the curriculum and teaching methods in algebra in order to inform discussions of any future revisions of the curriculum, taking into account evidence about future algebraic needs and current algebraic competence among different groups of pupils.*

With this brief we examined the influences on pupils' experience of algebra within all aspects of the pre-university education system. Separate sections of the report focus on:

- the pre-16 curriculum;
- the 16–19 curriculum—A-levels;
- vocational provision 14–19.

The report has been prepared at a time when there has been and continues to be an unprecedented political and media debate about the teaching and learning of mathematics, with blame being attributed to most aspects of the educational system, including the National Curriculum, teacher trainers, teachers and their qualifications, textbook writers and new forms of assessment. Within this report we examine (from the perspective of the teaching and learning of algebra) this multi-faceted system which constitutes mathematics education.

1.2 Perspectives from higher education

The working party was set up in response to the concern expressed by those in higher education about the mathematical background of their undergraduates. The observations of those in higher education were based on their incoming undergraduates, but the 'concern' was about the mathematical experiences of all school pupils. Accordingly the working party had aims and focus not only restricted to the background of potential undergraduates. Concerns in higher education were initially fuelled by university mathematicians and engineers.[1] In particular university mathematicians (Tackling the Mathematics Problem, 1995) identified a number of problems perceived by those teaching mathematics in universities: '(i) a serious lack of essential technical facility—the inability to undertake numerical and algebraic calculation with fluency and accuracy; (ii) a marked decline in analytical powers when faced with simple problems requiring more than one step; (iii) a changed perception of what mathematics is—in particular of the essential place within it of precision and proof' (p. 2). Physicists, engineers and those from other scientific disciplines are now actively concerned with what they perceive to be a serious problem. It has been suggested that as a result of undergraduates' difficulties with mathematics, university subjects such as biology and chemistry are becoming less and less mathematical (Stewart, 1996).

Students have always made mistakes with algebra. However, university lecturers are reporting that students with A-level mathematics are no longer fluent with symbols and often make algebraic errors which were previously characteristic of those students who were less well qualified in mathematics. For example, an undergraduate reading mathematics to degree level was happy to cancel terms inappropriately in the following expression:

$$\frac{a + \cancel{b}}{\cancel{b} + c}$$

and consistently made this error in examples of a similar form. This is but one example of the lack of symbolic fluency among these students. More importantly, university lecturers report that their students do not have any means of explaining and thus correcting these errors when they are pointed out to them.

One issue which needs examining is that not as many good students as the universities (including Oxbridge) would like opt to study mathematics at A-level and afterwards at university. As a subject, mathematics is competing with other disciplines to attract the best students. Some of these other disciplines (computer science, psychology, microbiology and economics, for example) have grown in popularity and now attract large numbers of the able and numerate students who, in the past, might have chosen to study mathematics.

Universities have been facing this competition for students at a time when they have been under pressure to recruit more students. Over the last decade the number of students entering universities has increased from approximately 12% to just over 30%. This has led to competition between departments and many universities have been forced to accept weaker students than they might wish in order to fill their places. At a time when there has been widespread concern over possible 'grade inflation' (Dearing, 1996), universities have not been able to raise the grades required for entry and some have even lowered the required grades. The *UCAS University and College Entrance Guide* gives the actual A-level points scores of the top and bottom 10% of their entrants. Although interpreting these is difficult (they do not seem to account for people who have taken an A-level in a previous year), they do suggest that students are often admitted with grades below the published offers. These figures also show that there is a large spread in the achievement of university entrants at A-level (see

[1] Numerous articles and reports have been published including: Barnard and Saunders, 1994; Tackling the Mathematics Problem, 1995; Howson, 1996; Sutherland & Pozzi, 1995; Mathematics Matters in Engineering, 1996

Appendix 1.1). Universities may not have worked out how to teach classes of students with such a large range of background qualifications.

The university situation feeds back into the school system through the graduates who are recruited into the teaching profession. The complexity of this dynamic system is one of the reasons why it does not make sense to attribute blame to isolated elements within it.

1.3 International perspectives

Exchanges between university students within Europe are beginning to highlight differences between the mathematical background of UK students and those from some other European countries. This seems to be particularly acute in engineering.

At the same time international comparisons focusing on pupils' performance on a range of test items have been used to investigate differences between pupils' knowledge of mathematics across different countries. The emphasis of these studies is whether our students reach the same standard as other students on international tests. These studies very rarely reflect or discuss the similarities and differences between the school mathematics culture in different countries. This culture will be influenced by such factors as the curriculum, the examination system, the school system, the training and qualifications of teachers, whether or not pupils retake years of schooling and the implications of this on the age of pupils.

Results of comparative studies are difficult to interpret because school mathematics is not a monolithic subject which can be unproblematically tested across cultures. It is questionable whether mathematics itself is culture free, but even if it were, school mathematics is not. This can be seen clearly in our discussion of the differences between assessment items given to 10-year-olds in France and England and 18/19-year-olds in Germany and England (Appendix 1.2). Research has shown that even small perturbations, such as changing the names of the letters used in an algebra problem, affect the ways in which students interpret a problem and thus their facility with the problem (Rojano *et al.*, 1996). For example, when asked to write an equation to convert hours *H* into seconds *S*, 17-year- old Mexican students were not able to interpret the letters *H* and *S* as variables because they were only familiar with using letters such as *x* and *y*. This was not the case for similar students in the UK, who could interpret the letters *H* and *S* as variables although they could not write a correct equation.

In addition many comparative studies may not be comparing like with like. In particular, school-leavers are often older in other countries (particularly Germany) than in the UK, although this is counterbalanced in part by the requirement to follow up to eight subjects throughout their school careers. It can also be misleading to compare the UK with other countries such as France, where mathematics plays a special role in the curriculum[2]: achievement in mathematics is used in France to establish who has the right to enter the celebrated *Grandes Ecoles*. There is therefore a clear incentive for all students who can cope with the *Bac S* (Maths and Physics) to take it even if they do not intend to specialize in mathematics afterwards. This is in contrast to the UK and Germany, where pupils tend to specialize according to their strengths and interests in order to maximize their credit (and their schools' credit) in the school-leaving examination. In many other European countries, and in particular in France and Germany, pupils repeat a year if they do not achieve the required standard.

Despite these limitations, the fact that studies are beginning to show that UK pupils are not as confident in and competent with some aspects of algebra as their counterparts in many comparable countries (NFER/SCAA, 1996) suggests that the issue ought to be probed more deeply. It is beyond the scope of this report to examine in depth the effects of these quite substantial cultural influences on the mathematics learning of groups of students. We urge that comparisons between countries take a more holistic view of the situation as opposed to concentrating on one aspect of the system on its own. Kimbell (1996) has pointed out that the recent discussion about the apparent success of Taiwanese schools concentrated almost exclusively on the importance of whole-class teaching but not, for example, on the fact that Taiwan spends 15% of its tax revenue on education, whereas the figure for the UK is 5%.

For the purposes of this report we have chosen to analyse a sample of assessment papers and examinations from France, Germany and the UK for pupils at the end of primary school, at the end of compulsory schooling and just before potential entry into higher education. We maintain that these questions give some indication of what is expected of pupils at these stages of education and also point to the differences in the school- mathematics cultures.

We include in Appendix 1.2 a description of the systems in France and Germany and some sample examination questions. Our analysis of a selection of these questions highlights the very different approach to school- algebra that has been taken in England and Wales, in comparison with France and Germany. The vast majority of pupils in France and Germany who stay at school to 19+ continue to study mathematics. The courses that they take require comparable skills in algebraic manipulation to those required in single mathematics A-level.

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[2] Personal communication with Claude Boucher, Chief Inspector of the European Schools, Brussels.

The examinations in France and Germany taken at 16+ play a slightly different role to GCSE. The Hauptschulabschluss and the Realschulabschluss in Germany are only taken by those who leave school at 15+ (about 30%) or 16+ (about 35%). They are not used as an intermediate level examination for the 35% of students in the Gymnasium who are going on to take the Abitur. In France, the Brevet, which is taken at 16+ by nearly all students (and passed by the vast majority),[3] is not usually considered important by mathematics teachers who teach mathematics for the Baccalaureate and do not usually put energy into preparing students for the Brevet.

What is striking about comparisons between expectations of 16+ pupils in France and Germany with those of pupils in England and Wales is that the vast majority of French and German pupils at this age are expected to engage with algebraic ideas which many of our students do not encounter until A-level (for further discussion see Appendix 1.2). Why do we have such different expectations of our students in the area of algebra than is the case in France and Germany? It seems to relate to a limited view of what pupils can be expected to achieve, which is almost always considered to be an inevitability as opposed to something that can be influenced by teaching. It is beyond the scope of this report to probe the cultural differences which might explain different expectations in different countries. However, these different expectations and related achievements do provide a counter-example to suggestions that only a small minority of students can succeed with mathematics.

1.4 Teaching and learning algebra

Traditionally pupils' introduction to school algebra was predominantly concerned with using and operating on literal systems. This is illustrated by the following excerpt from an algebra textbook (Fig. 1.1) which is similar to that which the majority of those of us writing this report would have experienced when we were at school:

6 Add $m^2 - 3mn + 2n^2$, $3n^2 - m^2$ and $5mn - 3n^2 + 2m^2$.
7 Add $3a^2 - 2ac - 2ab$, $2b^2 + 3bc + 3ab$ and $c^2 - 2ac - 2bc$.
8 Add $a^2b - 5ab^2 + 7b^2$, $2a^3 - a^2b + 5ab^2$ and $3b^3 - 2a^3$.
9 Subtract $3a - 4b + 2c$ from $a + b - 2c$.
10 Subtract $+ b - c$ from $c - a - b$.

Fig. 1.1 Example of algebra exercises from a 1950s textbook

This approach to teaching algebra involved presenting pupils with symbolic code as a means of generalizing from arithmetic in which 'letters stand for numbers'. The emphasis was on repeatedly practising algebra by working through a multitude of exercises.

The teaching of algebra remained essentially unchanged until the mathematical reform movement, starting in the 1960s and spearheaded in Britain by the School Mathematics Project. This movement reflected the changing role of algebra for mathematicians, placing an emphasis on algebra as a language for representing structures. Functions, mappings and the language of sets became a strong focus within many schools, and pupils' first introduction to algebra was now likely to be in the context of set theory.[4] Freudenthal maintains that the new mathematics had a disastrous effect on the teaching of algebra and was particularly critical of the ways in which letters began to be used to denote both sets and members of sets (Freudenthal, 1973). The effect of the new mathematics movement has also been criticized by Chevallard (1984), who maintains that what was lost within this reform was the dialectic between arithmetic and algebra which was so heavily prevalent in the earlier algebra textbooks. He stresses that this dialectic existed even before the construction of the algebraic language and that in Greek times there existed two arithmetics, a computational arithmetic (logistica) and theory of numbers (arithmetica).

In the 1970s and the 1980s considerable research evidence began to accumulate showing that the majority of pupils were not interpreting literal symbols in ways which were appropriate to algebra (Booth, 1984; Kieran, 1989; Küchemann, 1981). For example, pupils might think that a letter in algebra stood for its position in the alphabet or the name of an object (a for apple, b for banana). These results resonated with schoolteachers who had always found that school algebra alienated many of their pupils. In the UK, greatly influenced by this research, there began to be a shift in what constituted school algebra in the pre-16 curriculum with a substantial move away from the use of literal symbols (Sutherland, 1990).

In the UK the Cockcroft Report also had a considerable influence on changes to school mathematics and in particular school algebra. Recommendations related to 'understanding', situating mathematics within 'practical' problems and the need for a differentiated curriculum have all influenced the curriculum in ways which have resulted in less emphasis being placed on algebra. This is illustrated by the following quote, which draws some general conclusions about algebra, although the Bath Study referred to was carried out with a particular sample of non-university entrants only.

> *Formal algebra seems to have been the topic within mathematics which attracted most comment. Those engaged in the Bath Study were 'left with a very strong impression that algebra is a source of considerable confusion and negative attitudes among pupils'. In some cases this was because the work had been found difficult to understand: in*

[3] In 1995 90% of the age cohort born in 1980 took the Brevet examination and 73% of this age cohort passed the examination. A reform of the Brevet is due to give back some importance to this examination (personal communication with Antoine Bodin, Université de Franche Comte).

[4] In the UK the degree of abstraction in modern mathematics was less than on the Continent. See 'New Thinking in School Mathematics', Organisation for Economic Co-operation and Development (1969); 'Synopses for modern secondary school mathematics, OECD, 1st printing, June 1961.

> *other cases it was felt that exercises in algebraic manipulation and topics such as sets and matrices had had little point. (Cockcroft, 1982 para 201)*

As already discussed, school algebra has never been the same in all countries and has not evolved in the same ways in all countries over the last 10–15 years. We have had particular influences in the UK which appear to relate to our more pragmatic culture, although countries such as the USA, Canada, Australia and the Netherlands seem to be changing their school algebra curriculum in ways which are similar to the UK (see for example Treffers, 1993). In fact some of the current changes in the USA are heavily influenced by changes in the Netherlands and the UK. Countries such as France and Germany have also developed their school algebra curriculum over the last 15 years, but in ways that seem to be of a different nature to the changes in the UK. The situation is complex and there are even differences between school algebra in England and Scotland (Sutherland & Pozzi, 1995). However, no country other than England seems to have been so keen to delay pupils' experiences of the formal language aspects of algebra.[5]

The reforms to the algebra curriculum, already established in England and Wales, and currently taking place in the USA, are influenced by a desire to relate algebra to problem-solving. In fact algebra word problems which were traditionally used to teach algebra were always related to problem-solving. However, in England and Wales such word problems were considered to be too contrived and have almost disappeared from text materials, with an attempt being made to replace them with more relevant and realistic problems as recommended by the Cockcroft Report. Nowadays, problems involving patterns of tiles or matches (DIME, 1984; SMP 11–16, 1984) have become many pupils' first introduction to ideas of generalizing and formalizing. However, the algebraic purpose behind these questions is often lost with the teacher not knowing how to support pupils to move from their informal constructions to a formal and algebraic relationship (Stacey & MacGregor, 1997).

Sutherland (1990) has argued that one of the reasons why the algebraic language aspect of mathematics has become so under-emphasized in the UK relates to teaching approaches which place considerable emphasis on pupils' informal methods, making it difficult to teach the rule-bound aspects of the algebraic language. Thus it would appear that the algebra which pupils learn is inextricably related to the teaching approaches used. But if we consider teaching algebra to be similar to teaching a foreign language, there are many ways in which it can be taught that do not necessarily involve over-emphasizing rules. Brown and a group of secondary school teachers (Brown *et al*., 1990) developed ways of introducing the algebraic language to 11–16-year-old pupils which involved the teacher working with the whole class drawing on pupils' own awareness, but also transforming these awarenesses through use of the algebraic language. The aim was for students to experience the power of the algebraic language to support insight into mathematical structures.

Another reason why symbolic aspects of algebra have been under-emphasized is that it is clear that the mere use of algebraic literal symbols does not imply that pupils are acting and thinking algebraically.

Modes of algebraic activity

Within this report we shall use the phrase 'algebraic activity' to describe the kinds of encounters students ought to be having with algebra. Kieran (1996) has identified three components of 'algebraic activity':

> *Generational activities* which involve: generating expressions and equations which are the objects of algebra, for example, equations which represent quantitative problem situations (for example Bell, 1995); expressions of generality from geometric patterns or numerical sequences (for example, Mason *et al*., 1985); and expressions of the rules governing numerical relationships (for example, Lee & Wheeler, 1987).
>
> *Transformational rule-based activities,* for example, factorizing, manipulating and simplifying algebraic expressions and solving equations. These activities are predominantly concerned with equivalence, form and the preservation of essence.
>
> *Global, meta-level activities,* for example, awareness of mathematical structure, awareness of constraints of problem situations, justifying, proving and predicting, and problem-solving. These activities are not exclusive to algebra.

The nature of these activities is discussed more fully in Chapter 2.

We claim throughout the report that currently we prioritize generational activities in pre-19 education, to the detriment of transformational and global meta-level activities. The algebraic language is a tool which supports all of these activities. It had always been taken for granted that university students in such subjects as mathematics, engineering and many of the sciences would already be fluent in the use of this language before

[5] This is illustrated by the following excerpt from an internal discussion document written in January 1988 for consideration by the National Curriculum Mathematics Working Group. 'Very serious consideration needs to be given to the amount of algebraic manipulation which is now needed, and by how many pupils. The computer or algebraic manipulation calculator now makes it unnecessary for pupils to acquire much skill at manipulation such as factorisation and combination of algebraic fractions. It is not known how much (if any) pencil and paper facility is needed to understand what the computerised tools are doing and to understand the reasons for expressing algebraic formulae in a variety of forms. However, algebraic notation, such as that for powers will continue to be needed. Algebraic manipulation is an area in which profound change is likely in the next few years. At present it is an area where many pupils suffer severe loss of confidence. We believe it should only be expected in meaningful situations and when the case for it has been well made.'

entering university. However, fluency with symbols has become confused with rote manipulation. Rote manipulatation is often viewed in a pejorative way because it is seen as being in opposition to real understanding (Walkerdine, 1988). Thus fluency with symbols and all the attendant transformational rule-based activities have become under-valued.

When babies learn to talk they inevitably use words without understanding what they mean (as do adults when learning new conceptual areas). This is why a desire for pupils to understand mathematics before they use it gets in the way of the learning of the algebraic language. In other words, using the algebra language in incorrect ways is an inevitable and inextricable part of learning this language, which we argue ought to take place before students enter university to read mathematics, science and engineering. Implicit in much of the thinking behind the National Curriculum is the idea that all forms of mathematical representation are only add-ons to an already understood process. Gauss once said that what matters in mathematics is not notations, but notions (quoted in Stewart, 1995). We argue that the notion and the notation cannot so readily be separated, and good notation does facilitate thinking and communication in mathematics.

Tall (1996) stresses the importance of the use of symbols for compressing information in order to enhance thinking. 'I hypothesise that greater mathematical success comes not from remaining linked to the perceptions of the world through our senses, but through using the symbolism that is especially designed for *doing* mathematics and for *thinking* about it' (p. 28). He also emphasizes that 'mathematicians think powerfully precisely because they use the links *within* mathematics and do not relate constantly to the real world' (p. 30). Similar ideas have been discussed by Arzarello *et al.* (1997). Whereas it is important that students do link mathematics to the real world when appropriate, over-emphasis on this aspect is likely to detract from learning algebra. This is why the current emphasis on problem-solving is detracting from algebraic activity.

The issue for mathematics education is how to re-emphasize the role of symbols without precipitating a return to the traditional and often ineffective means of teaching algebra which were prevalent 20–30 years ago. These methods were ineffective because they only worked for a very small proportion of the school population and, as Cockcroft pointed out, actually alienated many pupils from mathematics. However, algebra and the algebraic language are central to mathematics and if we do not teach algebra then we are not teaching mathematics.

Arcavi (1994) has tackled the issue in a recent article. He identifies a number of aspects of symbol sense which include: an understanding of and an aesthetic feel for the power of symbols; a feeling for when to abandon symbols in favour of other approaches; and an ability to manipulate and to 'read' symbolic expressions as two complementary aspects of solving algebraic problems.

> *The solution of simple algebraic equations, as usually posed in standard texts and as usually taught in the classrooms, automatically arouses an 'instinct' for technical manipulation. Thus it requires a certain maturity to defer the 'invitation' to start solving, for example* $(2x + 3)/(4x + 6) = 2$ *and instead to try to 'read' meaning into the symbols. In this case, one might notice that, whatever x, since the numerator is half the denominator, this equation cannot have a solution. We claim that this* a priori *inspection of the symbols with the expectancy of gaining a feel for the problem and its meaning, is another instance of symbol sense.* (p. 27)

1.5 The impact of new technologies

Currently there are two main types of computer environments which are impacting on school algebra. The first is the interactive microworld-type of environment in which algebra-like symbols are used to approach quantitative situations, such as work with spreadsheets, graphics calculators and other programming environments (Noss, 1986; Sutherland & Rojano, 1993). Research has shown that when pupils are working in these environments they can learn to use symbols to represent general numbers, set up algebra-like expressions, work with the unknown and express relationships between variables. More importantly when working within these environments, pupils have to use a symbolic language so they are engaging with a range of rule-based transformational activities. Work with such environments also points to different ways to teach traditional algebra and emphasizes the importance of use-with-feedback in the learning of symbolic languages. When pupils work together at the computer they start using the computer-based formal language to talk to each other (Healy *et al.*, 1997). This oral work is an important aspect of learning a symbolic language.

The second type of environment is the computer algebra system (CAS) (for example Derive and Mathematica) which is now available on hand-held calculators, which the majority of post-16 students are likely to own by the end of the century. There is relatively little research on how CAS can be used for the teaching and learning of school algebra and yet there is considerable 'hype' and pressure from commercial companies. Independent research studies are urgently needed to investigate some of the claims being made. It has been suggested that CAS will promote a better conceptual understanding of mathematics: 'The use of a machine can be a benefit to a good conceptual understanding'(Algebra at A-level, 1996). The idea that seems to be implicit here is that because the machine is doing the symbolic manipulation the pupil will be free to develop understanding. It is not clear why this should be the case and relates very much to the previously discussed dominant idea that active work with algebraic symbols promotes rote as opposed to real understanding.

Arithmetic calculators can be used by pupils to avoid thinking about important mathematical objects (for example fractions) and to avoid thinking about structure (for example $a = b + c$ being the same as $a - b = c$). Computer algebra systems could help pupils focus on certain types of algebraic objects (for example equations), but as with arithmetic calculators they could detract from a focus on algebraic structure. The type of research question which needs to be asked is: Which algebraic objects and relationships are pupils using and thinking with when they work with a CAS, and how does this relate to the types of problems which is being solved? As with all aspects of algebra, the teacher will play an essential role in mediating the use of computer algebra systems. Calculators and computers are computational and not pedagogical tools (Pimm, 1995), although they can be harnessed by the teacher for pedagogical purposes.

Students can use computer algebra systems to solve all the traditional algebra manipulations which were the 'backbone' of school algebra 20 years ago. We need to analyse the purpose of these school algebra questions, which was certainly not to 'get the right answer', but to induct pupils into using and thinking with the algebra language. We maintain that this is still as important as ever within mathematics, engineering and many of the sciences.

The growth of IT has made it possible for students to manipulate many different types of external representations on the screen, involving symbolic, graphical and tabular forms. It is now possible to manipulate graphical representations in ways which were not possible on paper. Harnessing this new power within mathematics and school mathematics is the challenge for the 21st century.

1.6 Summary

Influenced by the Cockcroft Report mathematics educators have taken a particular approach to school algebra which prioritizes generational activities and pays little attention to transformational and global meta-level activities. This is characterized by:

- an emphasis on problem-solving related to 'real-world' problems;
- an emphasis on relating algebra to pupils' informal methods;
- a de-emphasis on the role of symbols.

The situation in England and Wales contrasts with the situation in other European countries, and in particular France and Germany. In England and Wales we expect less from the majority of pre-16 pupils in terms of learning algebra than is the case in France and Germany. In addition, the majority of students who stay at school to 19+ in Germany and France are taught algebra which is equivalent to that within A-level single mathematics.

Pupils can learn many important aspects of algebra, and in particular the idea of a variable, from work with interactive computer environments which use symbols to approach quantitative situations. However, we cannot emphasize enough that calculators and computers are computational and not pedagogical tools. Their use in schools will always have to be orchestrated by a teacher.

The teacher is crucial to all aspects of learning algebra because algebra does not relate to the real world and does not develop spontaneously within children. The teacher has to support pupils to make the leap from arithmetical to algebraic approaches to solving problems.

Throughout this report the changing situation over the last 10–15 years is examined. The influence of the National Curriculum, levels of attainment and Key Stage tests on what algebra/mathematics teachers are likely to teach is investigated in Section 2. The effects of changes in the pre-16 curriculum on algebra at A-level are the focus of Section 3. Finally in Section 4 the issue of algebra in the new vocational qualifications is addressed. This report was written throughout 1995/96 and we recognize that the system is continually changing. The report does not take account of changes made after August 1996.

2. The Pre-16 Curriculum

2.1 The changing situation over the last 15 years

After the Cockcroft Report (1982) changes were effected within mathematics education (Brown, 1996). New curriculum schemes were developed and in particular the new School Mathematics Project (SMP 11–16, 1984) which was ultimately used in more than 70% of schools. Primary and secondary DES-funded advisory teachers were appointed in each LEA to work alongside teachers. These were called 'Cockcroft' missionaries and the intention was that they would influence classroom practice as recommended by the Cockcroft Report. Curriculum and graded assessment schemes were developed for the lowest attainers in the 14–16 age group (for example LAMP, GAIM). Finally the new General Certificate of Education (GCSE) was introduced in 1986. 'Clear recommendations were given that the way to increase confidence and application skills is to broaden methods of teaching and assessment to include practical work, problem-solving, investigations and discussion, alongside the traditional exposition and practice' (Brown, 1996, p. 6).

With the introduction of GCSE the debate about process/content was highlighted. Specific curriculum materials were developed to support the GCSE examination which were intended to affect classroom practice, notably materials produced by the Shell Centre (1984). New approaches to

algebra were emphasized, for example generalizing from figurative patterns (see for example SMP 11–16, 1984). Algebra often became the 'hidden curriculum' within investigations, hidden in the sense that pupils were not necessarily aware that high marks would be awarded for generalizations expressed in algebra.

One effect has been that rote learning of algebraic manipulations and proof is being replaced by rote learning of pattern spotting (Coe & Ruthven, 1994).

The undoubtedly 'good ideas' embedded in many of the new curriculum materials started to become institutionalized in our school curriculum in quite unintended and unpredicted ways.

In 1989, the National Curriculum for Mathematics came into being and the arguments were renewed, with the introduction for the first time of specific process-related attainment targets for children up to 16 years old (Attainment Target 1: Using and Applying Mathematics).

The National Curriculum has changed at least three times since its original introduction. However, central to all versions of the National Curriculum is the idea of levels of attainment. Statements of attainment were organized into ten levels, on the recommendation of the TGAT report (DES, 1988). Kücheman (1990) has pointed out that the members of TGAT stated that they:

> *assume progress to be defined in terms of the national curriculum, and the stages of progress to be marked by levels of achievement as derived from that curriculum*. It is not necessary to presume that progression defined indicates some inescapable order in the way children learn, or some sequence of difficulty inherent in the material to be learnt. *Both of these factors may apply, but the sequence of learning may also be the result of choices, for whatever reason, which those formulating and operating the curriculum may recommend in the light of teaching experience.* (DES, 1988, p. 93; emphasis added)

Thus these levels were not overtly related to the ways in which children learn, or to any concern with mathematical coherence, but seem to have been a pragmatic response to setting clear age-related targets. Küchemann suggests that it is likely that the National Curriculum levels were influenced by the Concepts in Secondary Mathematics and Science (CSMS) research which classified items in selected areas of the mathematics curriculum into levels. 'As well as providing data on individual items, the CSMS work classified items in selected areas of the mathematics curriculum into levels and it is likely that this was seen as providing support for the use of levels in the National Curriculum, though this is not acknowledged by the mathematics working group' (Küchemann, 1990, p. 107).
Our analysis of the various versions of the National Curriculum suggests that the resultant 1995 version of mathematics has actually changed very little with respect to algebra. Even though the new Programmes of Study might appear to have changed, the level descriptions belie this and changes appear to be mostly cosmetic (see Appendix 2.1 for Attainment Target 2: Number and Algebra in 1995 National Curriculum).

Within this section of the report we examine the potential influences of the GCSE examination, the National Curriculum and textbook materials on teachers' practice.

2.2 Changes to school algebra

We maintain that the National Curriculum, national assessments and textbook schemes all combine to play an influential role on pupils' experiences of school algebra. In particular textbook schemes developed in response to the Cockcroft Report were used by the vast majority of pupils in the UK. Although we recognize that they were never used by all pupils and that their influence is decreasing, they do provide an indication of the culture of school algebra in the UK over the last 15 years.

Secondary school—Key Stages 3 and 4
Before we discuss (in Section 2.3) what we consider to be algebra and modes of algebraic activity for this age group, we describe here the main factors that are currently structuring what pupils learn.

Trial and improvement methods are a valuable technique for solving polynomial equations of degree 3 or more and a wide range of equations which cannot be solved algebraically. However, trial and improvement seems to be becoming the preferred and probably only method which the majority of pupils are confident with in pre-16 education. Interviews with A-level science students suggest that some of these students use this approach when solving simple linear equations (Sutherland *et al.*, 1995), which was not the case with a similar group of students in Mexico who were also interviewed. Vile (1996) found the existence of trial and improvement and systematic enumeration methods when a group of year 9 pupils were asked to solve a range of equations (see for example Fig. 2.1).

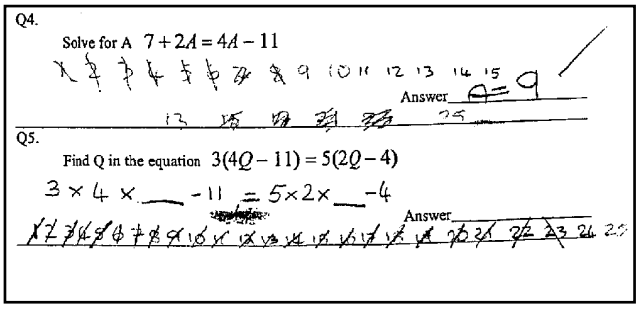


Fig. 2.1 Repeated trials to solve linear equations

That these methods have been found to exist among year 9 and year 12 pupils in England suggests that what could have been considered to be a spontaneous approach to solving simple equations is now becoming a taught method. More evidence for this is found in the Letts Study Guide, Key Stage 3 Mathematics (Williams, 1991, p. 74).

This approach to solving equations is explicitly examined at GCSE, often to the detriment of other methods, as illustrated by the following example (Fig. 2.2).

Judy is using 'trial and improvement' to solve the equation $x^2 + x = 11$

Complete her working and find a solution correct to one decimal place.

Try $x = 3.5$	$3.5^2 + 3.5 = 15.75$	too large
Try $x = 2.5$	$2.5^2 + 2.5 = 8.75$	too small
Try $x = 3.0$	.....	
Try $x =$	.....	

.....

.....

.....

Fig. 2.2 Solving quadratic equations using 'trial and improvement'

Solving quadratic equations by this method is not likely to provoke pupils to think about why there can be two, and only two, solutions to a quadratic equation or how this relates to the shape of the quadratic graph. Consider the following example from a GCSE candidate entered for the higher tier of GCSE and asked to solve the equation:

$$(x + 3)(x + 1) = 15$$

The pupil answered the question by saying,

'two numbers multiplied together give 15 ... must be 5 and 3. So $x = 2$ because $2 + 3 = 5$ and $2 + 1 = 3$'

The pupil appears to have developed an informal method for solving quadratic equations and is apparently unaware that this gives only one solution. In fact, this single solution is seen as the end of the process and s/he hasn't seen (−5 and −3), (1 and 15) and (−158 and −1) as alternative numbers whose product is 15, nor given any indication of being aware of the fact that none of the non-integer pairs whose product is 15 can give rise to a solution. Searching for one solution to a quadratic is actively encouraged by the type of trial and improvement approach, illustrated above, which is routinely examined at GCSE.

When pupils have become proficient with trial and improvement methods for solving equations they are unlikely to want to learn algebraic methods. In addition, the trial and improvement method may actually constitute an obstacle to the learning of algebraic methods. This is because trial and improvement involves working forwards from a 'known' starting number to the 'unknown' number, whereas algebraic methods involve working backwards from an 'unknown' number to a known number. The National Curriculum specifies that pupils should select 'the most appropriate method for the problem concerned, including trial and improvement methods'. This type of statement has institutionalized 'trial and improvement' *as* an algebraic method.

Problem-solving and process skills detracting from algebra. The National Curriculum has divided mathematics into four attainment targets. These divisions can result in any algebraic coherence being lost. Problem-solving is specified in the Number and Algebra Attainment Target 2 and much of the meta-level activities of algebra are specified in the 'Using and Applying Mathematics Attainment Target 1' strand. For example, level 7 of Attainment Target 1, 'pupils justify their generalisations or solutions, showing some insight into the mathematical structure of the situation being investigated' is arguably related to algebra, whereas problem-solving processes such as exploring number patterns and trial and improvement are specified in Attainment Target 2 (Number and Algebra). This has led to pattern-spotting almost becoming algebra to the detriment of other algebraic ideas. Moreover, classical algebraic methods for solving equations are not explicitly specified (in contrast to trial and improvement methods). In Section 2.3 we present a suggestion for showing, in a diagrammatic form, the relationship between different strands of an algebra curriculum.

Thinking algebraically involves compressing process into new mathematical objects, for example $3x + 7$. In algebra these new objects are objects to think with and not processes to be carried out (Tall, 1996). The new compressed objects maintain live connections with their related processes (Barnard, 1996). Currently our curriculum places too much emphasis on process, which becomes institutionalized as something to be taught, for example: Try some simple cases; find a helpful diagram; organize systematically; make a table; spot patterns; use the patterns; find a general rule; explain why it works; check regularly (Key strategies for investigations proposed in materials prepared by the Shell Centre (1984, p. 46)). This has led to the introduction of more empirical methods into our mathematics curriculum. These methods often inappropriately become the focus of teaching.

School algebra tends to be forced into spurious contexts in order to be taught and examined. The algebraic purpose of these contexts is not clear and the contexts are likely to detract pupils from focusing on algebraic activity. Our concern is that the purpose of the context from the point of view of learning algebra does not seem to have been analysed (cf, the use of a grocery shop context to introduce pupils to equivalent algebraic expressions in SMP 11–16 (see Appendix 2.2). Our point is that there is a difference between the use of context, such as those used in the traditional algebra word problems, and the current cosmetic 'dressing up' of a problem which appears to be merely aimed at motivating pupils to get started on a problem. We suggest that more research needs to be carried out on the

effects of 'realistic' contexts on pupils' mathematical learning and motivation. Publishers seem to demand this type of dressing up, often in the form of pictorial illustrations, and we question whether they are the best people to determine what mathematics our pupils are learning (see Pinel, 1996).

The use of symbols is often relegated to some later stage in the generalization process. Pupils are often expected, when writing up their investigations, to show that they have worked through the stages of: make a table, spot patterns, use the patterns, find a general rule. If a pupil does use symbols to express a general relationship without writing down some of the other stages teachers can often interpret this as showing a lack of 'real' understanding (Morgan, 1996). When emphasis is placed on generating and formulating algebraic expressions there is a tendency for not enough emphasis to be placed on using these expressions to some purpose. In other words, when symbols are produced they can be seen as the end point of a process and not the starting point for further thinking. In Appendix 2.3 we present what we consider to be a good example of pupils work in which symbols are generated in order to be used as a tool to think with.

National Curriculum level descriptions. Our analysis of the 1995 Key Stage 3 Mathematics test suggests that the idea of level descriptions has had the effect of straight-jacketing Key Stage 3 questions to fit the level descriptions for the various attainment targets to the detriment of the teaching and learning of algebra. Consider the opposite example from the 1995 Key Stage 3 test of levels 5–7.

The algebraic purpose of this question is related to the idea that it is possible to express in different, yet algebraically equivalent ways, the same pattern or sequence. So the question emphasizes that Sue and Owen have found different ways to express square patterns of dots. Sue has found a relationship which could be expressed as $n^2 = n + n\,(n-1)$. Owen has found a relationship which could be expressed as $n^2 = 2 \times n + (n-1)(n-2) + (n-2)$. However, in the assessment item the pupils are only asked to satisfy level 6 'When exploring number patterns, pupils describe in words the rule for generating the nth term of a linear sequence, questioning and checking the accuracy of their generalisation' and level 7 'Pupils express in symbolic form and explain the rule for generating the nth term of a sequence, where the rule can be formed by combining two linear functions'. They are actually provided with rules expressed in numbers and so have only to express in symbols someone else's rule. They are not asked to explain or justify why Sue and Owen's rules are equivalent.[6] Thus the whole potential of this type of problem has been lost by the need to fit it into the level descriptors. We recognize that these questions are developed for assessment and not teaching, but they may influence teachers who could view those who write assessment items as experts. Thus test items could destabilize teachers' confidence in what is mathematics and what is algebra.

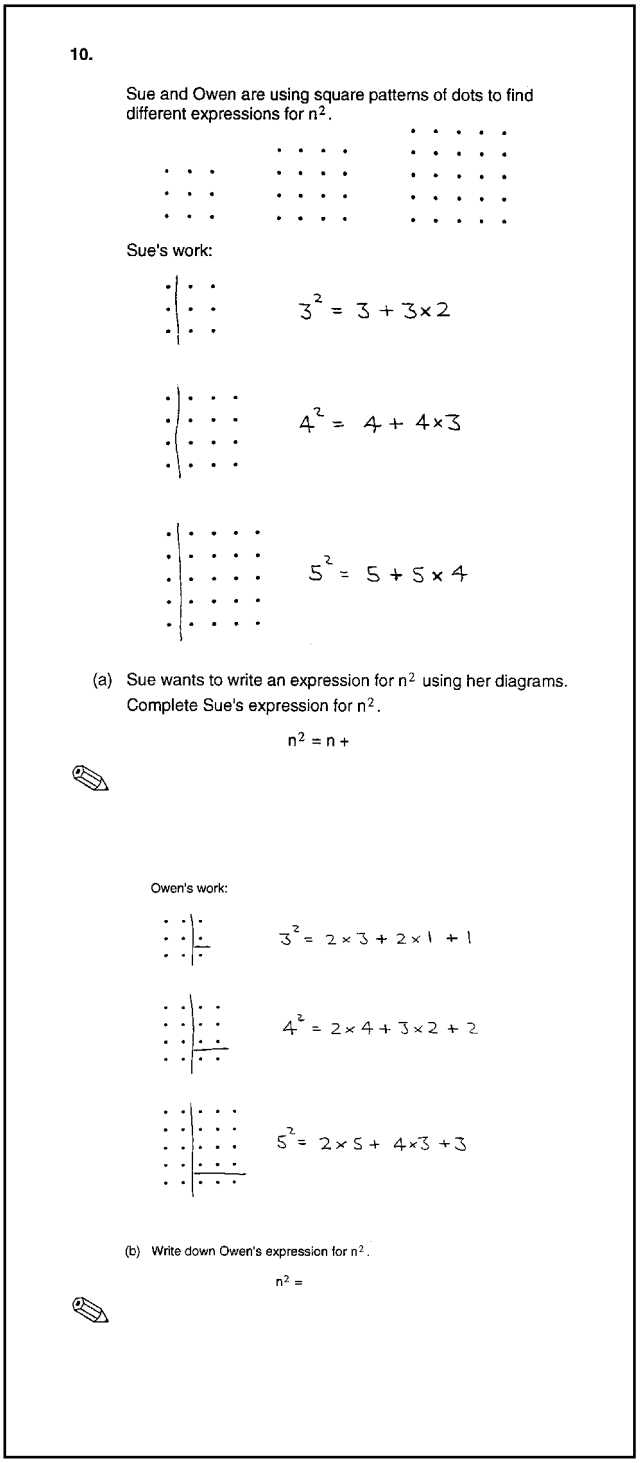


Fig. 2.3 1995 Key Stage 3 test item—level 5–7

The working group found many Key Stage 3 assessment items in which the algebraic purpose of the items was not clear (see Appendix 2.4 for an example of an item involving patterns of matchsticks in which the opportunity of assessing function and inverse function is not realized, presumably again because of the level of the question). As these levels impact on assessment items, which in turn impact on what teachers teach, algebraic coherence is being lost.

[6] An earlier version of this item did ask pupils to justify why the rules are equivalent but this was deleted after trials in the field, possibly because the question was too difficult for pupils. This deletion of the algebraic purpose of the question illustrates the way in which levels of attainment and Key Stage tests are having a cumulative effect on school algebra.

Mathematics is a complex conceptual domain. The attempt to reduce it to levels of attainment which can be tested at Key Stages 1, 2 and 3 with the possibility of some Key Stage 2 pupils being tested on algebra from level 6 and some Key Stage 3 pupils being tested on algebra from levels 3–5 leads to a quite 'senseless' fragmentation of what algebra is.

Moreover, whatever is specified in the Programmes of Study is then explicitly tested. This is illustrated by the Algebra Programme of Study (DES, 1995, para 4a, p. 17) 'explore number patterns arising from a variety of situations' which leads to the quite inappropriate proliferation of 'number pattern' assessment items and also the reference to trial and improvement in paragraph 5d with similar unproductive consequences from the point of view of teaching and learning algebra.

The influence of these test items on what teachers teach is likely to be substantial, within the current climate of league tables, market forces and teacher assessment. Moreover, student teachers are expected to reference levels of attainment in their lesson plans. Teachers could teach mathematics without paying attention to levels of attainment and Key Stage tests but these were actually set up to improve learning in schools, that is to influence teachers in what they teach.[7] However, if, as we have argued, the sense of algebra has been lost within the test items then this is likely to lead to pupils learning 'senseless' algebra. We believe that research is needed to investigate the effects of testing approaches and levels of attainment on what pupils learn, with a particular focus on algebra.

GCSE examinations

There is very little emphasis in GCSE examinations on algebraic methods of solving equations. The 'trial and improvement' method is usually examined, which is not the case for the algebraic method. Neill *(*1995) report that 'In GCSE papers the number of marks allocated for algebraic manipulations is low' (p. 2) and they go on to say 'It is possible for a candidate to attain a grade A in GCSE without addressing any of the algebraic items'.[8] An estimate of the total number of marks awarded for algebra in GCSE is 11% of the total marks.

Students can obtain a grade B in GCSE from the Intermediate Tier. This tier examines minimal algebra. This Intermediate Tier is targeted at grades C and D (available grades are B, C, D, E).

There appears to be a reluctance to present questions related to a more abstract use of algebraic symbols without dressing them up within a spurious context (for example 'Steve has found this diagram and expressions in an old book', MEG SMP 11–16, Paper 5, 1995) or the use of patronizing pictures

10 This formula can be used to calculate the percentage volume of carbon dioxide in the flue gases from a boiler:

$$V = k\left(1 - \frac{x}{21}\right).$$

V is the percentage volume of carbon dioxide,
x is the percentage volume of oxygen,
k is a constant for the type of fuel.

(a) Find V when $x = 4{\cdot}6$, $k = 11{\cdot}9$. [2]

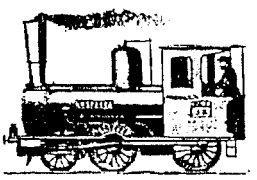
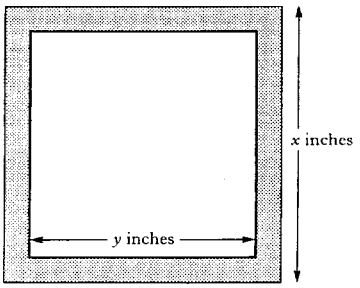


Fig. 2.4 Illustration 'bolted on' to question. MEG 1995 - Paper 5

to illustrate a situation as in the example presented in Fig. 2.4 from the same paper. Whereas we have found some appropriate uses of context in the Scottish Standard Level examinations (for example Fig. 2.5 below), English GCSE questions seem to prefer more artificially made-up questions as for example 'A possible points system for the high jump event in athletics is given by $P = a(M - b)^2$ where M is the height jumped in cm, P is the number of points awarded and a and b are positive constants' (Question 4, SEG, Higher Tier paper 6, 1994).



A square picture frame is shown above.

The border of the frame (shaded in the diagram) has uniform width and an area of 48 square inches.

(*a*) Show that $(x - y)(x + y) = 48$.

(*b*) Given that x and y are whole numbers each greater than 10, find suitable replacements for x and y.

Fig. 2.5 Appropriate use of context. Scottish Certificate of Education, Credit Level, 1993

Primary school—Key Stage 2

Some early aspects of algebra are now included in the primary school curriculum. Many primary schoolteachers are likely to have difficulties themselves with this area of mathematics and may not recognize the essential aspects of algebra (students can be accepted on courses training primary schoolteachers with a grade C at GCSE). This is likely to result in primary teachers either avoiding teaching algebra or teaching it in a mechanistic way.

[7] It is interesting to note that schools in the private sector do not legally have to teach the National Curriculum.
[8] We recognize that as from 1998, GCSE syllabuses will include a greater emphasis on algebraic skills.

The following example used by a primary teacher to teach that 5 x 3 = 3 x 5 illustrates how the particular choice of

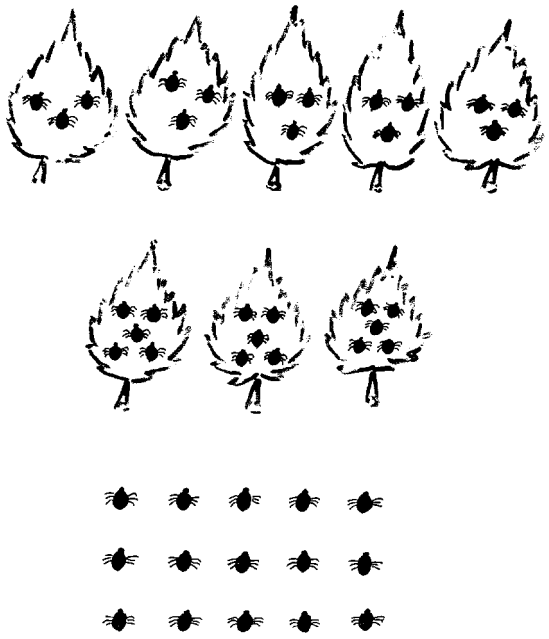


Fig. 2.6 Illustration used to teach 5 x 3 = 3 x 5

curriculum materials does not support teachers to emphasize the importance of structure in arithmetic/algebra:

A picture of 3 leaves, each with 5 ladybirds and another picture with 5 leaves, each with 3 ladybirds does not illustrate that 5 x 3 = 3 x 5. The pictorial representation used loses all the structural aspects of the relationship, the conclusion that 5 x 3 and 3 x 5 are equal relying solely on the fact that both are equal to the specific number 15 (obtained by counting) rather than on the underlying features which apply generally to *any* product of two numbers. This example illustrates how inappropriate use of pictorial images detracts from mathematics. However, had the first picture been of a rectangular array of dots with 5 rows each of 3 dots and the second one of 3 rows of 5 identical dots then this would be seen as structurally sound and containing the essence of the algebra. Many such curricula materials are influenced by pressures from publishers to illustrate mathematics in texts which they believe will sell. No account seems to be taken of the mathematical meanings which pupils are likely to derive from the presentation of a problem. More research needs to be carried out on curriculum developers' views on mathematics and whether they take into account the idea that the meanings pupils construct are inextricably linked to the materials with which they interact (Brousseau, 1997). In other words, the nature of the mathematics which pupils learn will be influenced by the specificities of the images which are presented to them.

Analysis of the Key Stage 2 test items suggests that not enough emphasis has been placed on the pre-algebraic notions of structure and general arithmetic. Consider the following item taken from the 1996 Key Stage 2 test (levels 3–5).

As illustrated by the pupil solution to this question, some pupils seem to be learning to use a 'trial and improvement' method for this type of problem. This strategy is likely to be influenced by work with calculators. In this way pupils are avoiding inverting operations, which are a precursor of algebra. Analysis of scripts from a recent study with British and French primary school pupils[9] shows that the British Key Stage 2 pupils made much more use of written trials on paper than was the case with the French pupils. This result has to be treated with caution as the French pupils may have been reluctant to write down their 'rough' working. However, there were no French examples of 'informal' strategies such as tallies or repeated addition instead of multiplication.

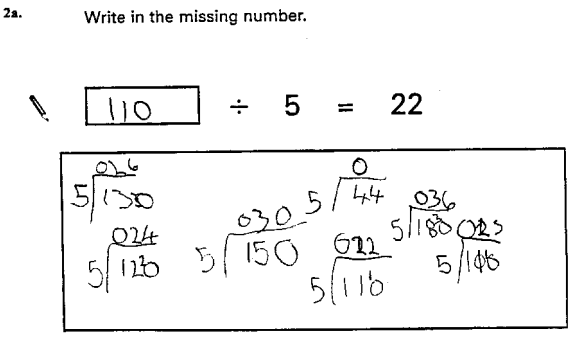


Fig. 2.7 Example of pupil's work from 1996 Key Stage test

This emphasis on 'trial and improvement' methods, which we have found within the curriculum from Key Stage 2 throughout the pre-16 curriculum, is possibly one of the most worrying aspects of the curriculum from the point of view of developing algebraic ideas and relates to an over-emphasis on answer as opposed to method. More research needs to be carried out on whether these 'trial and improvement' methods do constitute an obstacle to the development of algebraic ideas.

Even when the calculator is not allowed in the test (as was the case in one of the Key Stage 2 papers in 1996) this is not likely to affect the ways in which the pupils solve this type of problem. As illustrated in the above pupil's work some pupils use a mental or paper 'trial and improvement' method when tested on this type of question. From an algebraic perspective it is not a question of mental or calculator strategy, but which mental or which calculator strategy. It seems as if the seemingly 'good ideas' for calculator use in the primary classroom did not take into account the ways in which these would impact on the development of algebra. It is not so much the use of calculators at primary school which is of concern,

[9] These scripts were collected as part of a project 'Being a Pupil in England and France: findings from a comparative study', Osborn *et al.* (1996).

but their use in conjunction with certain types of problems and methods of solution. This again relates to the role of the teacher.

Currently there is an over-application for primary PGCE teacher training. The evidence from those recently accepted on one primary teacher-training course suggests that only approximately one tenth of these applicants have studied a mathematics-related degree with almost none of these with a degree in mathematics alone. We suggest that with such over-application it should be recommended that a much larger proportion of these potential primary teachers should have studied a mathematics-related degree. We recognize that this recommendation needs further investigation and that the possession of a degree in mathematics does not guarantee effective teaching of mathematics in primary schools. We also recommend that primary schoolteachers should have a higher qualification in mathematics than grade C at GCSE, which could involve studying mathematics to the age of 18/19 in line with recommendations made in the Dearing Report.

To support existing primary teachers we recommend funding for in-service training on the lines of the GEST Grant 3 20-day courses which were positively evaluated by Ofsted.

2.3 Algebra and modes of algebraic activity

As discussed in Section 1.4, one way of organizing algebraic activity is to use the following subheadings: *Generational activities; transformational rule-based activities; global, meta-level activities.* These are elaborated below. However, from the point of view of the curriculum they cannot be separated. It is particularly important that teachers encourage global meta-level activities as pupils work on generational activities, otherwise their algebraic purpose will be lost.

(i) *Generational activities* which involve:

- generalizing from arithmetic;
- generalizing from patterns and sequences;
- generating symbolic expressions and equations which represent quantitative situations;
- generating expressions of the rules governing numerical relationships.

When working on these activities the emphasis should be on meta-level activities, such as awareness of the relationship between general rules and the structure of the problem, relating pattern to problem, predicting and justifying.

(ii) *Transformational activities* which involve:

- manipulating and simplifying algebraic expressions to include collecting like terms, factorizing;
- working with inverse operations;
- solving equations and inequalities with an emphasis on the notion of equations as independent ‘objects’ which could themselves be manipulated, working with the unknown;
- shifting between different representations of function, including tabular, graphical and symbolic.

These activities are concerned with equivalence, form and the preservation of essence. They require an appreciation of the need to adhere to well-defined rules and the notion of mathematical expressions as objects in their own right. It is necessary to build up through repeated exposure to these mathematical objects, a wider symbol sense in which pupils gain a feel and intuition for the ways in which such objects relate to each other.

(iii) *Global, meta-level activities* which involve*:*

- awareness of mathematical structure;
- awareness of constraints of the problems situation;
- anticipation and working backwards;
- problem-solving;
- explaining and justifying.

These activities transcend all of mathematics.

Fig. 2.8 presents an overview of the essential elements of a pre-16 algebra curriculum and the interrelationship between these elements.

Foundations for early-algebra
We have identified a range of activities which we consider to be precursors to algebra and which should take place both in the primary school and secondary school. The activities here are characterized by:

(a) Expressing relationships involving more than one operation. These rules could be expressed in words or symbols. If symbols are used they should be viewed as a succinct form for expressing the rules.

(b) Working with inverse arithmetic operations.

(c) Working with relationships expressed in number patterns: forming and continuing terms in a number sequence.

(d) Working with relationships expressed in figurative patterns: copying, continuing, devising, recording, describing in words and translation from words back to patterns and predicting.

(e) Focusing on language as a means of precisely expressing relationships.

(f) Awareness of mathematical constraints.

2.4 The impact of new technologies

As discussed in Section 1.5, new technologies will inevitably impact on pre-16 school-mathematics. Work with interactive environments such as spreadsheets, programming languages and graphics calculators has shown pupils can learn the symbolic aspects of algebra as a language of communication. This points to ways of introducing algebra as a symbolic code through use in communication. Within certain computer environments the grammar of the algebraic language can be

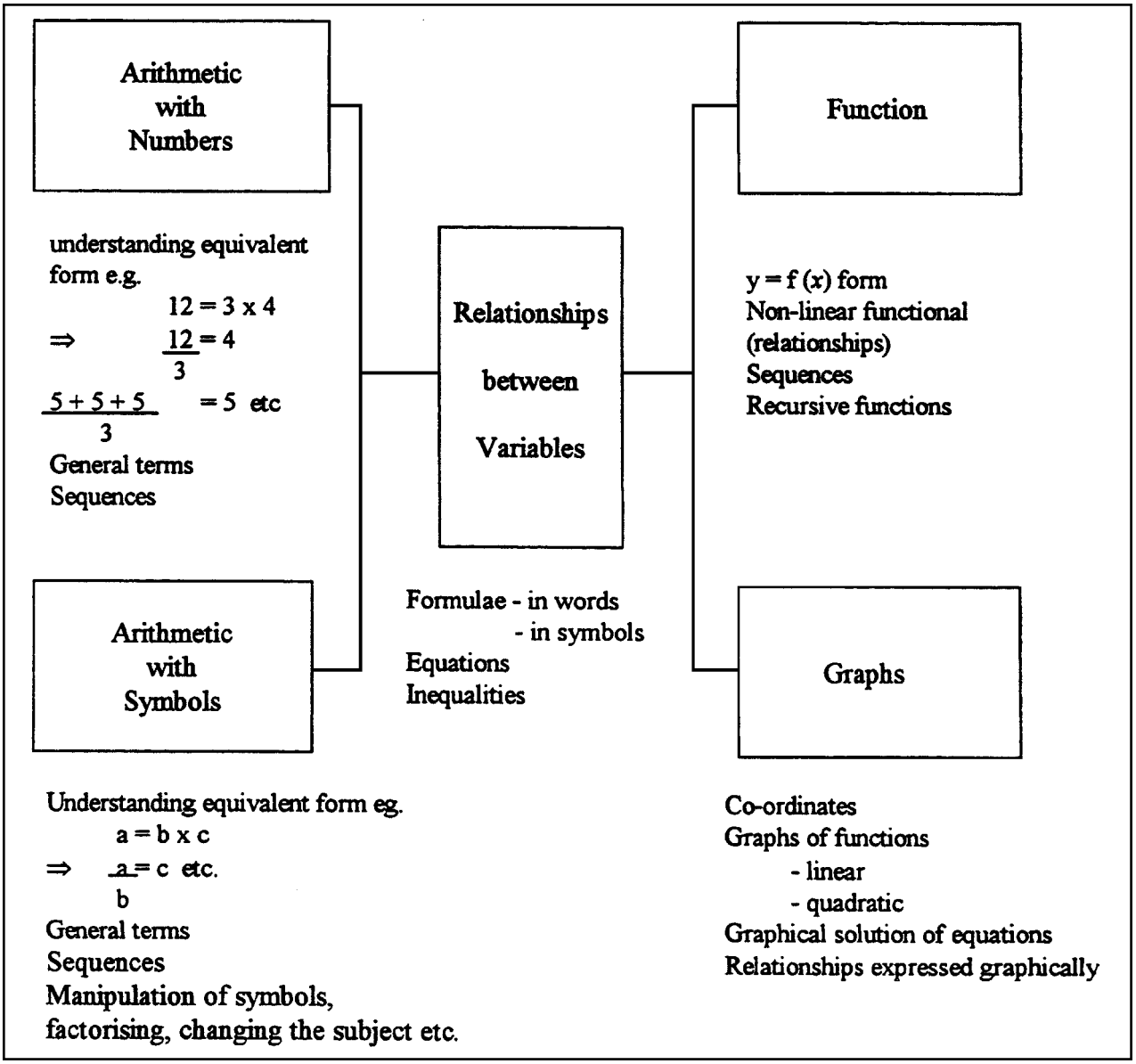


Fig. 2.8 Diagramatic view of algebra for pre-16-year-old pupils

learned through feedback from the computer. This is easier to do with computers than paper because the computer will not allow you to enter incorrect syntactical expressions. In addition pupils are likely to be more resistant to feedback from a teacher telling them about syntactical errors than to feedback from a computer.

However, it is possible for the teacher to organize classroom activities so that pupils do receive feedback on the way to use the algebra language, through, for example, group work in which communication on paper is a focus, or whole class discussion orchestrated by the teacher.

Within this report we have claimed that the ways in which 'good' curriculum ideas such as investigations and exploratory work with calculators become institutionalized in the curriculum produce learning effects which their advocates would not have predicted. This is likely also to become the case with the use of computers. So, for example, work with spreadsheets could become transformed into an ineffective (from the point of view of learning mathematics) institutionalized practice. This implies the need for ongoing research and development as opposed to the launching of a new curriculum idea which is expected to radically change everything which was previously considered 'bad' practice.

It is mathematics and not the technology which should determine where the teaching emphasis is placed. The arrival of algebraically competent technology could change the ways in which we teach. Also tied to these points is the issue of how we then assess work with new technologies. We urgently need more research in this area.

2.5 International comparisons

As discussed in Appendix 1.2 analysis of test items and curriculum materials used with 10–11-year-old pupils and 15–16-year-old pupils in France[10] and Germany has shown that:

- French and German primary school pupils are not being explicitly asked to use trial and improvement methods to solve problems.
- French and German pupils at GCSE equivalent are expected to reason through whole questions and do not have the questions broken down into steps as is the case in GCSE.
- French and German pre-16 courses place more emphasis on the rule-based transformational aspects of algebraic activity than is the case in England and Wales.
- The French Brevet places emphasis on mathematical structure in that pupils are asked to present some answer in a particular form as opposed to being asked to complete the process of computation.

2.6 Summary

In conclusion, we maintain that the current algebra curriculum for the under-16s, is in need of substantial overhaul. The National Curriculum was an attempt to improve the teaching and learning of mathematics in England and Wales. We maintain that in the case of algebra, this particular form of National Curriculum with the attendant Key Stage testing and publishing of league tables has not had this desired effect. We make the following recommendations:

1. More emphasis needs to be placed on the learning of the algebraic language in the pre-16 curriculum. As with natural language, learning the rules of the algebra language occurs through extensive use and feedback on incorrect use. This does not imply a return to the 'traditional' routine approaches to introducing the use of symbols, but it does imply extensive use through class discussion and written work over time.

2. The National Curriculum is currently too unspecific and lacking in substance in relation to algebra. The algebra component needs to be expanded and elucidated and also reorganized. The group were especially critical of the woolly and all-encompassing wording of statement 3c of the Algebra Attainment target for Key Stage 3 and 4 on page 15 of the current Mathematics National Curriculum which states:

   Pupils should be given opportunities to: (3c) manipulate algebraic expressions; form and manipulate equations or inequalities in order to solve problems.

3. The levels of attainment within the National Curriculum work against the teaching and learning of algebra. The structuring of the National Curriculum should take into account both the need to preserve mathematical coherence and a consideration of how pupils learn algebra. The notion of levels on which this National Curriculum is predicated needs critical appraisal. For this to happen mathematicians and mathematics educators have to work together.

4. There is a need to emphasize the meta-level reasoning aspects of algebraic work and to make this part of teaching. Insufficient attention is currently given to this aspect. Pattern-spotting is in danger of becoming a new orthodoxy which all too often does not appear to have direct links with algebraic structure. Future publications concerned with the teaching and learning of algebra for pre-16-year-olds should make the need for such links a priority.

5. There is a need to see greater credit and recognition given in all assessments related to algebra to the reasoning and argument elements and consequentially less credit to any 'final answer'. However, aspects of problem-solving should not be inappropriately turned into objects of assessment (for example, 'pattern-spotting' should not be explicitly assessed).

6. Too little is made of the opportunities for recognizing the rich connections between algebraic processes and other subjects. The use of generalized statements and/or descriptions in other areas of mathematics and other subjects which are then interpreted to create new perspectives should be encouraged, but the group also recognizes that work is needed to identify these so as to help teachers and learners to maximize their experiences. Algebra should not be seen in isolation from these other subjects. The group urges that resources be made available to allow this identification work to be done.

7. Teachers of younger and later developing children need support and guidance in recognizing the essential nature of algebraic structure. Work needs to be done to develop materials and courses to achieve this and so the group recommends that resources be made available to facilitate this. In particular, funding should be made

[10] Personal communications with Claude Bodin suggests that French mathematics examinations are beginning to embed questions in more 'realistic' situations as a result of current reforms.

available for in-service teacher training.

8. Too little attention is given to work on the idea of algebraic equivalence. We recommend that a higher priority is given to such work so that learners become much more familiar with recognizing equivalent statements such as:

$$3 \times 4 = 12 \text{ and } \frac{12}{3} = 4$$

9. Interactive symbolic computer environments (for example spreadsheets and graphic calculators) offer new ways of introducing algebraic ideas to pre-16 pupils. In particular they support teachers to introduce algebraic ideas such as variable and working with symbols. However, teachers have to teach pupils to make links between computer-based and paper-based work. Also it has to be recognized that it is not the tool, but how the tool is used that is important.

3. The 16–19 Curriculum: A-level

In England and Wales there are two main routes to higher education, the A-level and the vocational route. Whereas A-levels have been the traditional entry point to higher education, increasingly more students are entering via the vocational route. This section of the report is concerned with the teaching and learning of algebra within A-level courses. Vocational courses are discussed in Section 4.

3.1 The impact of GCSE and the National Curriculum

As discussed in the previous section, in the stages leading to GCSE more emphasis should be placed on algebra, algebraic thinking, symbolization and manipulation. Global meta-level activities, such as finding structure, justifying and proving, have received minimal attention at the pre-16 level. There is also a strongly held view that the concept of proof is to some extent unnecessary, because its place has largely been taken over by investigative work. As emphasized in the previous section, investigative activity only makes sense mathematically if it does include this aspect of justification. The levels within Attainment Target 1 do not encourage teachers to include this type of meta-level activity in their teaching of more open-ended problems.

Questions on GCSE papers are often framed in such a way that they actually discourage the algebraic approaches that would be the most effective way of tackling a problem. Students are often explicitly asked to use 'trial and improvement' methods to solve equations and marks are only obtained if this method is used. Pupils are rarely explicitly asked to use algebraic methods. GCSE questions which test algebra are often presented in an inappropriate 'realistic' context which again detracts from the algebraic aspect of a problem.
GCSE questions tend to structure a problem in such a way that a pupil rarely needs to use initiative to decide *how* to solve a problem. The current assessment trend of sub-dividing questions within GCSE does not encourage anticipation and thinking backwards which are two subtle aspects of algebraic activity.

Moreover, the National Curriculum levels can have the effect of restricting a pupil's experience of algebra before entering an A-level course in that the most substantial algebraic ideas are specified in the higher levels and teachers tend to delay the teaching of these ideas until previous levels have been achieved. Our argument is that many of these algebraic ideas (for example: level 8 'pupils manipulate algebraic formulae') should be introduced at an earlier stage of the 11–16 curriculum.

When GCSE was first introduced, recruitment to A-level mathematics was expected to be from pupils who obtained grades A or B. Nowadays students with a grade C can also be accepted on A-level courses. A recent study has shown that in 1994 of all those (22,693) pupils who attempted A-level mathematics after a grade C at GCSE only 17% obtained a good grade (i.e. C or higher), and 59% did not achieve a grade at all (William, 1996, p. 41).

The situation is even worse for pupils who have studied the Intermediate Tier at GCSE, because this tier contains hardly anything which could be called algebra. Yet pupils who achieve pass grades at this tier of entry can now form a sizeable proportion of the pupils in an A-level class. Pupils with a B grade at GCSE could have studied this Intermediate Tier. This was not possible when GCSE tiering was first introduced in 1988.

This issue is becoming compounded in several ways. It has been suggested that one effect of league tables is that many schools are not entering even their most competent pupils for the Higher Tier of entry (Dearing, 1996). This is because a pupil has a better chance of passing with a C or B grade when entered for the Intermediate Tier and so schools 'play safe'. From the A-level perspective this leads to a disastrous situation. Recruitment from the Intermediate Tier has led to a particular problem for FE colleges and sixth-form colleges, because pupils enter these colleges with a certificate simply indicating the grade achieved but not the tier of entry. Thus their GCSE grade does not provide any indication of their competence in algebra.

A study by the JMC and SCAA (Brown, 1996), which was set up to investigate the step between GCSE and A-level in mathematics, points out that 'Both teachers and examination scripts attest to the fact that candidates can obtain a GCSE Grade A with few manipulative algebraic skills, although algebraic skills are essential and fundamental at A-level'.

3.2 Entry requirements for A-level

Nowadays many pupils cannot cope with the algebraic demands of A-level mathematics without some sort of 'extra' algebra. Currently this 'extra' algebra is being incorporated in a number of different ways throughout A-level courses. We believe that this situation is not satisfactory because with this approach pupils are less likely to develop adequate algebraic confidence and competence before entering higher education. In addition other aspects of the A-level curriculum receive less attention as a result of having to insert the algebra which was previously in O-level into the A-level course (for example, solving quadratic equations).

We maintain that there are only two possible alternatives to this 'algebra gap' between GCSE and A-level mathematics.

The pre-16 curriculum should incorporate more algebra than that set out in the National Curriculum. The issue still remains about whether this should be 'algebra for all' or only for those who will need it within post-16 courses. However, more and more pupils are progressing to post-16 courses which require confidence and competence with algebra. In this section of the report we are only concerned with pupils who progress to academic A-levels, but similar issues arise with those pupils who enter vocational courses as will be discussed in Section 4.

Post-16 institutions should develop 'bridging' algebra courses for some students before they start A-level. MEI have recently had validated the Foundations of Advanced Mathematics Course, which credits students with bridging the National Curriculum gap from level 7 (the level many students have achieved at GCSE) to level 9 which is the more appropriate level from which to start an A-level course in mathematics. There are problems with this approach: How much time will a student spend on the A-level programme? Can it be done in the traditional two years? How does a teacher cope with the vast achievement range now spread from A* at the Higher Tier to C at the Intermediate Tier? Is such a notable range class fair to any of the students in it? If the weaker students are separated from the rest and given special attention are they seen as non-starters before they even begin? If additional support is to be provided where will the funding come from and will other courses suffer as a result of funds being reduced to support this activity? Clearly, there is a whole range of issues here for SCAA and the awarding bodies to consider.

The Dearing Review has made the recommendations that there ought to be bridging courses between GCSE and A-level (Recommendation 133) and a new GCSE short course in additional mathematics, limited to grades A*–C, which should be taken by all students who wish to do A-level mathematics (Recommendation 132). This idea of an additional GCSE in mathematics might appear to be a powerful one because it provides an incentive for the more able students and could actually help boost recruitment for AS- or A-level further mathematics. Nevertheless, thought needs to be given by SCAA to this recommendation. Its implementation could actually backfire badly. As is the case with tier of entry at GCSE, the effect of league tables could be that if schools do not have to enter their students for this new examination, many will not and so students who could succeed at it will be denied the opportunity to do so. This would have the reverse effect to that intended: it could further decrease the numbers doing A-level mathematics. It is also a particularly British 'elitist' solution when compared with the pre-16 mathematics experiences of students in France and Germany (see Appendix 1.2 of the report). In fact in France the Brevet examination (taken by pupils of a similar age to the GCSE) has very little importance and is not used as an entry point to the Baccalaureat. As was intended to be the case in the independent school system, students who were going on to study A-level were not entered for an O-level in that subject, because teachers were teaching for A-level and viewed the O-level examination as a potential saddle-point in achievement. Moreover, constructing another examination hurdle for A-level does not address the need to incorporate more algebra in a diffused way throughout the 11–16 curriculum.

3.3 The changing A-level scene

We discuss below the main changes to the A-level curriculum which are affecting the teaching and learning of algebra.

Changes in attitudes towards algebra

Much of the algebra content of O-level has now entered into the first year of an A-level course. However, when algebra is introduced in A-level it is often camouflaged and may not even be explicitly called algebra. We suggest that this relates to a desire to de-emphasize algebra within the school mathematics curriculum, as already discussed in Section 1.4 of the report. For example, in the MEI course, the chapter dealing with introductory algebra is called 'The Tools of Problem-Solving'. All the other chapters in the book have names which refer to their mathematical content, for example Co-ordinate Geometry, Trigonometry and Integration. In the new Nuffield A-level scheme algebra has been camouflaged by the increasing amount of words which surround it and by a decrease in the number of practice examples which students are asked to carry out. For example in Chapter 1 of Book 3, Equations and Inequalities, the algebraic language is surrounded by so much text in English that it would be difficult for a student to get a visual feel for an algebraic exposition. As is the case with other new A-level schemes (for example SMP 16–19) the use of graphical methods receives more attention at A-level than it would have done 15 years ago

and this has resulted in a decreasing emphasis on algebra. The issue is a question of balance as graphical methods are also valuable.

Changes in the student populations

Many more students stay at school to study A-levels than was the case 15 years ago. The number of those who choose to study A-level mathematics relatively decreased (as a proportion of number of students studying A-level) and absolutely decreased between 1985 and 1995.

Year	No. of students studying A-level mathematics
1985	88,000
1990	69,500
1992	66,459
1993	59,010
1994	56,000
1995	56,695
1996	61,442

A-level mathematics has been changing partly as a result of attempts to encourage more students to study the subject. (The absolute number of students increased from 1995 to 1996 and it is too early to say if this is a general trend.) This is resulting in a different 'client' group compared to 15 years ago.

Changes to the nature of school mathematics

These are similar to those discussed within the pre-16 curriculum; that is, more emphasis on problem-solving, and open investigational approaches. There is more emphasis on statistics and data handling, but less emphasis on such topics as co-ordinate geometry, geometry and mechanics. Mechanics, for example, used to provide students with opportunities to practise and become technically fluent in the use of algebra (Williams, 1996) which is not the case with statistics. Many of these changes can be traced to the recommendations in the Cockcroft Report on the role of contextualizing in mathematics. This is reflected in the A-level examination questions as illustrated by the 1995 SMP 16–19 examination, taken at the end of a two year course (Appendix 3.1). Questions tend to place more emphasis on the generational activities of algebra; that is, on the construction or presentation of equations which represent quantitative problem situations (for example, Questions 2 and 3). In many cases when students have to construct equations for themselves they are provided with considerable support (for example, Question 5). There is minimal emphasis on the transformational activities of algebra. The aim of contextualizing problems within realistic situations can become contrived (for example, Question 6) and often gets in the way of a potential algebraic treatment of the problem.

Changes in the content of A-level

One of the key features of the present curriculum is the Common Core for A-level mathematics. This was first introduced in the early 1980s and has since undergone several revisions. The core accounts for 50% of the A-level syllabus and 70% of the AS syllabus.

At each revision some topics have been edged out to make way for additional material and it is claimed that students have to be familiar with a wider range of mathematics than ever before. This is true to a limited extent, but it gives a false picture. Looked at over time, the removals from the syllabus far outweigh the additions. Geoffrey Howson, in the Gresham College seminar 'The Mathematical Ability of School Leavers' (December 1996) cites changes to the new A-level syllabuses for 1996 and tabulates to what extent the syllabuses (including non-core material) now cover material that was in the previous core but not in the present core (see Appendix 3.2). There is a marked variation between the different examining boards here, sufficiently so to be worrying.

The losses are considerable. Going back ten years or so the list of deletions would include much more, including many topics that are important from an algebraic point of view. Whatever the reason for removing these topics from the curriculum, the resultant effect is a de-emphasis on topics which were likely to inculcate an elaborated symbol sense. Some examples of A-level question papers from 1986 are included in Appendix 3.3. Those questions marked with an asterisk (*) would not be possible in today's papers on the grounds of syllabus content alone. The issue of what students had to know in order to pass these examinations is complex. Current examinations are supposed to test what students know. However, it does not make sense to divide mathematical knowledge up in such a reductionist way and it is not clear that the current approach is any more effective in testing what students know than the previous approach.

The Dearing Review recommends that the A-level Common Core needs to be reconsidered and that SCAA and the awarding bodies should enter into discussion to determine what such a core should be. SCAA is to draw on the advice of the Mathematics and Science Consultative Group that was set up by SCAA in March 1996 with the approval of the Secretary of State. In discussing the Common Core it is important to ask :

> *To what extent does the Common Core contribute towards a coherent and relational understanding of mathematics?*
>
> and
>
> *To what extent does the Common Core provide a coherent body of knowledge which will be of value for students pursuing degree courses in maths, science or engineering (or any other subject with a broad-based mathematical content) and also to those who will use their knowledge of advanced mathematics in employment?*

In Section 3.5 we discuss a possible way of restructuring the Common Core.

Changes to the nature and extent of algebra being examined in A-level

In comparing A-level mathematics questions from current (1994 and 1995) examinations and those from ten years ago we were struck by the fact that there is almost as much variability between boards as there are changes over time from the point of view of what algebra is being examined (see Hirst, 1996 for further discussion of this issue). Some examinations represent new courses which did not exist ten years ago (for example Oxford and Cambridge's examination for the SMP 16–19 course and Oxford's examination for the Nuffield course). There is also the issue of modularity which we discuss later. For some examination boards there has been little discernible change (for example NEAB). However, some of these examinations are more influential than others in terms of the numbers of students who enter them. Overall the following conclusions can be made:

Some boards *lead students through examination questions at A-level in a step by step way* and provide support which makes it unlikely that they will attempt to get a sense of the whole question and work in an anticipatory manner, which is so important in algebra. This is illustrated by the following question Fig. 3.1 taken from the MEI Specimen Paper 1994. Part (i) of this question tells the student to complete the square, so that they do not have to work this out for themselves. They are then *provided with information* so that they do not have to think for themselves about the structure of the completed square form and work backwards to find the value of 'a'.

(time allocation 15 minutes; 20% of paper total)

(i) Show that $x^2 + 4x + 7 = (x + 2)^2 + a$, where a is to be determined.

(ii) Sketch the graph of $y = x^2 + 4x + 7$, giving the equation of the axis of symmetry and the co-ordinates of the vertex.

The function f is defined by $f: x \Rightarrow x^2 + 4x + 7$ and has as its domain the set of all real numbers.

(iii) Find the range of f.

(iv) Explain with reference to your sketch, why f has no inverse with its given domain. Suggest a domain for f for which it has an inverse.

Fig. 3.1 Excerpt from MEI A-Level Specimen Paper 1994

As the following solution shows, the first part of the problem can be solved by working forwards to find a.

$$\begin{aligned} & x^2 + 4x + 7 = (x + 2)^2 + a \\ \Rightarrow\ & x^2 + 4x + 7 = x^2 + 4x + 4 + a \\ \Rightarrow\ & 7 = 4 + a \\ \Rightarrow\ & a = 3 \end{aligned}$$

$$\therefore\ x^2 + 4x + 7 = (x + 2)^2 + 3$$

Many questions on the *pure mathematics papers emphasize applying mathematics to practical situations* (see for example questions from SMP 16–19, 1995 in Appendix 3.1) which was not the case ten years ago. Algebra and applying mathematics to practical situations are both crucial aspects of mathematics education—but they are distinct aspects. Using algebraic symbolism to model a practical situation is not algebra.

Students are often *provided with subtle support* when working with the transformational rule-based aspects of algebra. Consider, for example, question 9 of SMP 16–19 (Fig. 3.2). In part (b) of this question pupils are presented with support with factorizing the cubic in terms of being asked to find the coefficients of the quadratic equation (a, b and c).

9 The function $f(x) = 3x^3 - 11x^2 - 95x \div 175$ has 3 linear factors.

(a) Find $f(5)$ and use your result to explain why $(x - 5)$ is **not** a factor of $f(x)$.

(b) The function f(x) may be written in the form

$$f(x) = (x \div 5)\ (ax^2 \div bx \div c).$$

Find the values of a, b and c and hence write $f(x)$ as the product of its three linear factors.

(c) Find the values of x for which $f(x) \geq 0$.

Fig. 3.2 Question 9 from SMP A-Level 1995

Some *questions in current A-level papers would previously have been examined within the old O-level* (for example, Questions 2 and 3 in London P1, Appendix 3.4 and Fig 3.3, and Fig 3.4). What was examined in the old O-level varied between boards, but overall a greater degree of algebraic fluency was required in O-level than in the Advanced Tier of GCSE (Sutherland & Pozzi, 1995). Examples of questions from O-level papers are given in Appendix 3.5.

2. The straight line passing through the point $P(2, 1)$ and the point $Q(k, 11)$ has gradient $-5/12$

(a) Find an equation of the line in terms of x and y only.
(b) Determine the value of k.
(c) Calculate the length of the line segment PQ.

Fig. 3.3 Question 2 from London A-Level P1, May 1995

3. Show that the elimination of x from the simultaneous equations

$$x - 2y = 1,$$
$$3xy - y^2 = 8,$$

produces the equation

$$5y^2 + 3y - 8 = 0.$$

Solve this quadratic equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied.

Fig. 3.4 Question 3 from London A-Level P1, May 1995

These changes to the examination papers are resulting in less algebra being examined with a likely resultant effect on what is being taught at A-level.

Changes to the assessment mechanisms of A-level mathematics

Many of the recent changes are being made to make examinations more student-oriented (modularity, shorter examinations on a reduced syllabus, shorter questions, more structured questions almost telling the students what to do, more contextualization, more helpful formula books and so on).

Many syllabuses have become modular. This is a controversial area because there are differences in the ways in which boards have approached modularity. For some, but not all boards, modular papers present students with shorter questions and do not examine topics to the same depth as before.

The examinations tend to be of shorter duration than before (typically 1–$1^1/_2$ hours for each paper). Linear syllabuses still have longer exam papers ($2^1/_2$–$3^1/_2$ hours). There are serious discrepancies between the boards here. ULEAC pure mathematics module papers last $1^1/_2$ hours and students, in June 1996, had to attempt ten questions on module P1 and nine questions on module P2. The MEI module papers, on the other hand, all last for $1^1/_4$ hours and students have to attempt only five questions. Within the UCLES modular examination, each module finishes with a long question and overall, for this board, students now take 9 hours of examinations as opposed to the previous 6 hours.

For the standard A-level mathematics (in which the pure mathematics accounts for half the syllabus and applications accounts for the other half), the ULEAC students have to take two pure mathematics modules with a total examination time of 3 hours; the MEI students have to study three modules and a comprehension paper (lasting up to one hour) in pure mathematics, giving a total written examination time of up to $4^3/_4$ hours, and they also have to study coursework. The SMP scheme is different yet again, as is the AEB scheme.

However many teachers report the positive effects of modularity. Early feedback in the A-level course can be very important for students who might otherwise have ‘wasted’ the first year of the 6th form; weaker students can more readily manage the smaller chunks of mathematics required for a module which appears to be having an effect on the number of students studying A-level mathematics (evidenced by the increase in numbers in 1996); the integration of AS- with A-level is easier in the modular system; the potential exists to link chunks of post-16 mathematics with other qualifications, such as GNVQ.

As with many educational reforms in England and Wales we do appear to be ‘experimenting on the job’ . Our concern here is not with modularity as such, but with the extremely variable approach to what constitutes a modular examination. This suggests that more research is needed in order to understand the effects of form of assessment on the nature of mathematics which is being assessed.

The impact of new technology
As is the case of the pre-16 courses, A-level is responding to new technologies. The new Nuffield A-level, for example, incorporates the use of graphic calculators throughout the course as does the SMP 16–19 course. The influence of these new technologies changes the emphasis on the mathematics taught. For example more emphasis on graphical methods can result in less emphasis on algebraic methods, although this does not have to be the case. New technologies have made mathematical modelling more accessible at A-level. Although it can be argued that this is a very valuable part of the new A-level, it must be recognized that without an adequate algebraic background modelling will always be limited to extremely simple situations.

Computer algebra systems are making the situation even more complex. Do they imply that more or less algebra is needed? This is discussed in Section 3.8.

3.4 A-level further mathematics

The number of students taking A-level further mathematics is now approximately 5,500 as opposed to 13,000 in 1980 (although there has been a slight increase over the last two years). This has had an effect on the recruitment of students on to university degree courses in mathematics. In the past, it was assumed that students for mathematics degrees would enter university with a double award in mathematics at A-level. Nowadays, students are often accepted on to such courses with only a single mathematics qualification at A-level. This is undoubtedly an effect of the Cockcroft Report (1982) which recommended that 'We hope too that those who select students for admissions to Higher Education will recognise that there are sound educational as well as economic reasons for offering only single-subject mathematics at A-level and will not put either direct or indirect pressure on schools which have only limited teaching resources in mathematics to provide double-subject courses, especially for students to whom it is not well suited' (para 588).

The universities have adjusted their courses to use the first year as a levelling course to take account of the differing backgrounds of their students, and many have initiated four-year degrees. Nevertheless, as the recent report 'Tackling the Mathematics Problem' (1995) makes clear, higher education is far from happy with this current situation.

The Dearing Review proposes that more students should be encouraged to study either a full A-level further mathematics or to go half-way and study an AS-level in further mathematics (Recommendation 135). This is an important suggestion that should be followed through with positive action. Many schools and colleges are finding it difficult to support study of further mathematics because of the small numbers of students involved. Students usually do further mathematics as a subject beyond their normal three-subject choice of A-levels. Funding, at least from the Further Education Funding Council (FEFC) which funds all colleges of Further Education and all Sixth Form Colleges, is problematical; the FEFC allocates only a fraction of its normal A-level funding unit per student for a fourth A-level subject. This does not cause problems where the student does a non-mathematical subject as the fourth subject because the student does this subject in a class that already exists. However, with further mathematics a special class has to be created. Small numbers of students are therefore not viable as independent groups.

A further problem with further mathematics is that its take-up will not fit in readily with Dearing's proposal for the National Advanced Diploma award. Students who opt to study further mathematics will be disadvantaged because they will not be able to generate the additional breadth of study to qualify for the award.

3.5 Algebra and modes of algebraic activity at A-level

This report is about algebra. Interpreting what constitutes algebra is its main focus. As with the pre-16 curriculum, the A-level curriculum should be concerned with the following modes of algebraic activity namely (i) generational activities, (ii) transformational activities and (iii) global meta-level activities (Kieran, 1996). At A-level, algebra becomes predominantly symbolic in nature. A-level mathematics students need to be able to 'think with' symbols.

Also at A-level the content of algebra cannot be separated from much of what is considered to be pure mathematics, for example, calculus and trigonometry. In addition, algebra as a tool is used within every other more 'applied' branch of A-level mathematics including statistics and mechanics.

If the Common Core is to be revisited, as the Dearing Review proposes, then instead of using content as a means of defining the Core, perhaps the all-important criterion should be modes of algebraic activity. As stressed above these should permeate aspects of content such as co-ordinate geometry, functions, sequences and series, trigonometry, exponential and logarithms, calculus, etc. Below we outline what we consider to be the desirable algebraic activities that should be built into this approach:

- developing fluency in algebraic manipulation and transformation;
- developing facility with notation and understanding the importance of notation;
- working to precise definitions, learning the meaning of mathematical language;
- awareness and appreciation of the importance of mathematical structure;
- developing flexibility of viewpoint and a fluency to transform to different representations;
- using symbolic forms as mathematical objects with an existence of their own, independent of any explicit physical reference;

- doing and undoing—the significance of inverse operations;
- awareness of definitions and constraints;
- the importance of generalization;
- establishing relationships;
- the importance of precision and elegance in mathematical exposition;
- the role of both deductive and inductive logic;
- concepts of proof;
- the development of a critical facility, so that students can expose the fallacy of an illogical argument or paradox;
- reading mathematical texts and communicating mathematical ideas.

Using these ideas it would be possible to construct a pure mathematics syllabus which included this within all the key areas of importance—numbers (including complex numbers), sets, binary operations (including concepts of commutativity, associativity and distributivity), functions (including polynomial, rational, circular, exponential, logarithmic and any other suitably defined functions), graphical representations, series, limits, Cartesian geometry, parametric forms, calculus, vector algebra and geometry, linear algebra, and other topics—which would help to bind the subject together in a coherent manner.

3.6 International comparisons

England and Wales have long had an intensely specialized curriculum for post-16-year-olds. This has led to the view that somehow we produce a better, more knowledgeable group of students at the age of 18 or 19. But what is the evidence for this?

Many academic students in England and Wales do no mathematics beyond GCSE, even though they may have obtained a very high grade in mathematics at GCSE. These students have chosen to do three combinations of humanities or arts subjects for A-level. There is some evidence, cited in 'The Take-Up of Advanced Mathematics and Science Courses' (NFER/SCAA, June 1996) that an increasing number of students is mixing A-level mathematics with two non-scientific subjects. The Dearing Review could have recommended a compulsory broadening of A-level studies so that everyone had to study mathematics through to the end of schooling, but it did not do so.

In France and Germany, for example, students follow a much broader curriculum than here and the vast majority of students have to study mathematics until they leave school. (see Appendix 1.2). This allows for more informed decisions about choosing degree courses, or employment, which involve mathematics in one way or another. The main conclusions from our comparisons of pre-university courses in England and Wales, Germany and France can be summarized as:

- The examinations that specialize in mathematics in Germany (Leistungskurs) and France (Bac S) are more demanding algebraically than further mathematics A-level.
- The manipulative algebraic skills required in single A-level mathematics are comparable with those in the non-mathematics specialist pre-university qualifications in Germany (Grundkurs).
- The Grundkurs requires significantly greater skill in formulating problems algebraically than does single mathematics A-level.
- Mathematics is compulsory for the vast majority of 16–19 students in Germany and France, not just those specializing in mathematics (although in Germany not all of these students are assessed on the basis of the written examination).

Whereas we recognize that our A-level examination assesses a wider range of mathematics than similar examinations in France and Germany, the issue here is about how important it is to teach and assess algebra. It is possible to criticize the German and French examinations for placing too much emphasis on formal mathematics/algebra. However, the difference in approaches is now so striking that we maintain that our shift away from algebra at A-level is a cause for concern.

The International Baccalaureat makes for another interesting comparison. Universities in the UK used to be very keen on the IB, but nowadays it is harder to get good grades in the IB than in A-level mathematics and consequently schools are finding that less students are choosing to study for the IB. Appendix 3.6 shows currently agreed International Baccalaureat/A-level equivalences. Again league tables are the driving force behind which examination a student studies as opposed to educational considerations.

3.7 Teaching and ways of learning

Teaching time for A-level is being cut back and there is an increasing tendency to teach to the syllabus, and specifically to the examination of each particular syllabus. This teaching to the syllabus is also reinforced by fractioning the subject up into modular units. One issue which needs to be analysed more fully is the potential fragmentation of algebra through the modular approach. The module breaks sometimes appear arbitrary and inhibit the natural development of the subject matter. Very often the extension of a topic will take place in a later module. The essence of mathematics is to encapsulate and generalize. At the very moment when the teacher would like to do this the constraints of the syllabus prevent it. This is similar to the issue of the straight-jacket effect of levels of attainment at the pre-16 level, discussed in Section 2.

The teacher is central to an effective delivery of A-level mathematics, because he or she acts as a catalyst for discussion and the clarification of ideas; the teacher helps to establish unexpected connections between different topics

and helps the students to refine their methods of presentation and to use notation effectively and succinctly. However, many institutions are now moving away from teacher-directed lessons to student self-paced learning. The justification for this approach is that larger numbers of students can be accommodated in one class and that this is no bad thing given the spread of achievement on entry. The more likely explanation is that this is yet another cost-saving exercise. Textbook schemes at pre-16 which embraced individualized approaches (for example SMP 11–16) are now moving away from this approach. Why then should individualized learning at A-level be considered to be desirable?

Open-ended projects at A-level encourage students to pursue their own line of enquiry. This can often work against the algebraic and more structural aspects of mathematics. Collection of data and pattern-spotting tend to become the dominant activity in ways which are similar to what is happening with mathematical investigations at GCSE level. For example the following question was set as an open-ended investigation as part of the SMP 16–19 modular A-level scheme.

> The triangular numbers are 1, 3, 6, 10 etc., i.e. $\frac{1}{2}n\,(n+1)$
> The square numbers are 1, 4, 9, 16 etc., i.e. m^2
>
> Which numbers are both triangular and square? The first two are 1 and 36.

We have analysed one student's response to this question (the student received a grade B at A-level). The student's solution can be characterized as being an empirical exploration of the problem (using a spreadsheet) with numerical verification of specific cases. Even in showing how one triangular number leads to the next in his empirically found list, he did not show that this process always produces a square triangular number from a given one. He did not attempt any form of mathematical proof. We recognize that what one student produced cannot be generalized but it does show the existence of a phenomena which needs further investigation. Our analysis of this problem shows that in fact at A-level an empirical approach is the only one possible, because other treatments require mathematical knowledge which is only available at undergraduate level (see Appendix 3.7 for a discussion of a possible solution). We can only conclude that empirical approaches, as opposed to algebraically related proof, is what is intended to be taught and examined.

We support the idea of coursework within A-level but are concerned that some coursework is not encouraging the transformational and global meta-level aspects of algebraic activity. In particular, justification and proof should be an integral part of open-ended investigational work at A-level.

3.8 The impact of new technologies

Computer algebra systems are a very recent innovation which could have far reaching effects on A-level mathematics. Some research is beginning to show that when these tools are used by experienced teachers they can support students to develop an understanding of variable and function (Heid, 1996). However, we believe that it is important in the UK to take a serious and considered approach to what can be learned with computer algebra systems, how this relates to teaching and the type of problems being solved, and not rush into inappropriately incorporating their use into the new A-level courses.

We must be careful not to suggest that because CAS can be used to carry out the manipulative aspects of problem-solving, that this will inevitably provoke better mathematical understanding. As discussed throughout the report, algebraic activity is intra-mathematical and not related to the real world. Students need to be able to think with symbolic objects and it is very likely that they learn to do this by manipulating these objects for themselves. So a computer environment which carries out the manipulations for them could detract from a development of the symbol sense which we have been advocating throughout the report. For this reason many members of the group were concerned with some of the comments about the benefits of CAS which have been made in the recent NCET document, 'Algebra at A-level' (1996). More research is needed to investigate some of the claims being made.

What will be crucial will be how CAS are used and which problems students are asked to solve. An emphasis on proof at A-level will become even more important. The challenge is to find constructive uses for CAS that actually enhance the teaching of mathematical concepts. Mathematics should drive the use of technology and not vice versa. As with all computer work students' approaches will tend to be empirical and we need to question how this empirical work relates to theory and proof.

One possibility might be to use CAS to generate conjectures and provide plausibility checks that can then be verified by analytical methods. Exploration of genuinely difficult problems, such as piecewise algebraic matching of arbitrarily drawn curves, could be encouraged through constructive use of CAS. CAS might also be used to help students develop different strategies for solving problems, and any tool that helps do this is beneficial.

The purpose of an A-level mathematics course is to teach important mathematical concepts. The reasons for this are numerous, but they should include the fact that mathematics is a logical system; that mathematics can make precise, definitive statements; that mathematics is both an inductive and a deductive system of reasoning; that mathematics is interesting in its own right; that mathematics harnessed to other subjects has tremendous analytical and predictive power.

It is the mathematics itself, and not the technology, that should determine where the teaching emphasis is to be placed.

Some examination boards have tried to introduce examination papers which explicitly make use of CAS, but SCAA has put these initiatives on hold as it is felt that they are premature. Coursework options might be the appropriate place in which CAS could be introduced.

3.9 Assessment

Current assessment practices in mathematics place more emphasis on correct answers than on the process of solution, which is the main emphasis of algebra. In what way can examinations, of any kind, pay enough attention to the processes of reasoning as opposed to the obtaining of the correct answer?

Students should be judged on their ability to evaluate critically each link in a chain of reasoning. Faulty reasoning could provide one sort of framework for probing their understanding as illustrated by the following sample question:

We wish to solve:

$$\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$$

$$\frac{x+5-5(x-7)}{x-7} = \frac{4x-40}{13-x}$$

$$\frac{4x-40}{7-x} = \frac{4x-40}{13-x}$$

$$7-x = 13-x$$

$$7 = 13$$

Comment on the reasoning and expose the fallacy.

Students could be tested on their symbol sense as in the following questions ‘Take an odd number, square it and then subtract 1. What can be said about the resulting numbers?’ (Arcavi, 1994).

Ideas about assessment cannot be separated from mathematics itself. Mathematics has to lead assessment and not vice versa. As discussed throughout this report different ways of dividing up and presenting questions substantially changes the mathematics being assessed.

It may well be that students’ competence and confidence with algebra could be improved dramatically if more attention were to be given to new forms of syllabus design and assessment. This is a difficult task. To be carried through successfully it would need collaboration between all interested parties across the entire educational spectrum, including teachers, teacher trainers, academics and representatives from the awarding bodies and SCAA.

3.10 Summary

A multitude of factors which include new forms of assessment, the introduction of computers and an emphasis on making mathematics more relevant have worked together to nudge out the transformational rule-based and global meta-level aspects of algebra from the A-level curriculum. It is not clear that this was intentional, but given the evidence that this is the case we cannot be surprised by the concerns expressed by those in higher education.

Nowadays many students have to devote much valuable time to the development of algebraic and manipulative skills at the start of any A-level course, and time throughout the course has to be found for students to continuously work on these algebraic activities. Given this situation it is likely that many A-level mathematics students never become confident and competent with algebra. Our analysis of the situation at A-level leads us to make the following conclusions and recommendations.

1. Students should not be encountering the manipulative aspects of algebra at A-level, for the first time. In the stages leading to A-level more emphasis should be placed on algebra, algebraic thinking, symbolization and manipulation. The pre-16 curriculum should incorporate more algebra than that set out in the National Curriculum.

2. Post-16 institutions should develop ‘bridging’ algebra courses for some students before they start A-level.

3. Algebra at A-level should not be camouflaged. If there are difficulties inherent in the subject then students will have to face these difficulties.

4. Problem-solving in ‘real’ contexts provokes students to work in procedural ways in an attempt to find a quantitative answer as opposed to the intra-mathematical activity of algebra. In particular, ‘open-ended’ problem solving at A-level can result in students developing an empirical approach to mathematics. When working algebraically students have to suspend for a while a need to relate to the problem situation.

5. Many current A-level examination questions provide students with more support with the algebraic aspects of a problem, or are less demanding algebraically than was the case 10–15 years ago.

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[11] Some students also currently follow ONC/OND courses, or other vocational qualifications (Appendix 4.1).
[12] Broad equivalencies are described in the Dearing Review (1996, Table 4, p. 8).

6. The extreme variability between the emphasis on algebra in different A-level examinations is likely to result in students entering higher education with variable experiences of algebra.

7. The vast majority of non-mathematics pre-university students in Germany and France are expected to be competent with manipulative algebraic skills which are equivalent to those experienced by students studying a single A-level mathematics.

8. The incorporation of CAS into A-level should be treated with caution until more is known about what mathematical ideas students are likely to learn from working with these environments.

9. Content should not be separated from modes of algebraic activity in the A-level common core.

10. More attention should be given to syllabus design and assessment in order to promote algebraic activity at A-level.

4. Vocational Provision: 14–19

4.1 Introduction: growth of vocational education

As the impact of expanded provision beyond the end of compulsory schooling grows, increasing numbers of young people and mature students are studying applied courses through General National Vocational Qualifications (GNVQ), or vocational courses through National Vocational Courses (NVQ).[11]

For the purposes of this report we have concentrated on provision through General National Vocational Qualifications, which may be awarded at Foundation, Intermediate or Advanced level.[12] All GNVQ students must pass the core unit 'Application of Number' (at the appropriate level—1, 2 or 3). Engineering GNVQ students must also study mandatory mathematics specific units. There is an optional mathematics unit for GNVQ science students. For many courses some mathematics is inherent within mandatory or optional units (e.g. Unit 8, BTEC 'Advanced Health and Social Care', units within BTEC 'Advanced Construction and the Built Environment').

This section deals with a rapidly changing educational domain and because of this we have structured the section in a different way from the others. Recommendations are made throughout the section and when we feel that there is not enough known about an issue we call for further research.

The focus of this part of the report is the mathematical-algebraic needs of such 'vocational' students. We consider the relevance of algebra and pre-algebra in respect of their:

- daily life-skills and numeracy (Section 4.2);
- competence in employment and supporting vocational/GNVQ study (Section 4.3);
- access to higher education from advanced GNVQ courses (Section 4.4).

4.2 Life skills: numeracy and algebra/pre-algebra

We suggest that it is important for all students to become confident and competent with certain areas of pre-algebra and algebra in order to function effectively as citizens within a society which is increasingly shaped by mathematics and where problems are increasingly converted to forms for which there is a calculable solution.

The ability to interpret and formulate problems expressed symbolically is one which is increasingly assumed, not least the ability to set up a spreadsheet to perform desired calculations, a task which is standard now for many administrative staff.

In its earliest expression, this may mean developing an understanding of the structure and form of number, the ability to acquire what has been commonly called 'number-sense' or a 'feel for number'. So, for example, Evans and Thorstad (1994) discuss the need for school governors to understand aspects of percentage and issues such as rate of inflation and the ways in which their informal practical daily life knowledge influences their interpretations within a school budgeting context.

At a more advanced level this would include the ability to extend such awareness and proficiency to more abstract symbolically expressed forms. As technology becomes more widely available, this does not necessarily imply a proficiency with paper and pencil techniques of 'traditional algebra', rather a competence with the sense of algebra and its use in expressing relationships. For example, the ability to describe the relationship which exists between girth and height of a tree and volume of useable timber and the fact that these might be used to calculate numerical solutions to a given problem such as estimating the potential timber production from a given plantation.

Indeed, the growth of IT has led to a greater presentation of data in visual format, which includes tables and a range of graphical representations. This makes it important for the public at large to be able to interpret critically these representations.

We maintain that the focus of 'algebra for citizenship' ought to be one which emphasizes a feel for number structure and pre-algebra, which encourages critical analysis of 'what is being said' and which uses appropriate symbolism within specific contexts, including IT. It should not focus on formal

algebra.

4.3 Competence in employment and supporting vocational study

Algebraic demands of the applied vocational routes and NVQ

There is a division between the technical vocational fields which requires a significant algebraic component and others which may require very little in the way of formal algebra. Whichever route they choose, all students enrolled on GNVQ courses must complete the compulsory core skills units.

The 'Application of Number' core units (levels 1, 2 and 3) do not explicitly specify algebra within the categories which constitute the range specifications. However, within the Advanced level unit there are some discernible algebraic ideas specified within 'Number'. For example:

- descriptive interpretation of pattern and relationships using words at lower levels and symbols at level 3 (e.g. yield per acre = harvested crop divided by seed expressed as y = H/S);
- construction and interpretation of linear graphs;
- use of simple formulas to calculate numerical values including the ability to find the value of terms other than the subject of a formula and hence have some awareness of structure of expressions.

We have two major concerns here. The first is that we do not consider that this type of 'hidden' algebra is adequate. This omission of algebra when number, shape, space and measures, and handling data are named and included is not dissimilar to the lack of emphasis on algebra within pre-16 courses and a reluctance to use the name algebra in many new A-level mathematics textbooks. Our second concern is that the current mechanism for assessment of the core skill, mathematics, through work in other mandatory units is inadequate. GNVQ guidelines emphasize that:

> *Evidence should show that the student is confident in the use of each technique listed*
>
> *the skills should be learned and demonstrated through activities which will enhance students' capacity to perform effectively in vocational settings.*

Many students accepted on GNVQ advanced courses have weak mathematics backgrounds. It is almost impossible for them to learn (and for teachers to teach) the necessary mathematics when it is embedded in complex vocational settings.

Within the Foundation and Intermediate level 'Application of Number' units there is no discernible mathematical-algebraic content. This does not bode well for progression to advanced GNVQs, particularly in the more technological

[13] Results from an ongoing research project suggest that it is very unlikely that GNVQ Science students will learn crucial aspects of mathematics for science when it is delivered entirely within the vocational science context (Molyneux & Sutherland, 1996).

[14] In a recent project (Molyneux & Sutherland, 1996) only 11 out of 23 Advanced GNVQ science students had obtained higher than a grade C in GCSE mathematics before entering the course.

areas. It is also reasonable to assume that those seeking employment, having gained an intermediate GNVQ, might need to switch careers several times during their lifetime.

The lack of a mathematical-algebraic base to their GNVQ study is likely to limit progression to more skilled, and technologically creative work, restricting opportunities to use of existing technology within their field, rather than being sufficiently mathematically literate to engage in modification and supplementation of material.

In conclusion we recommend that algebra becomes a more explicit part of GNVQ courses. In particular we suggest that:

- *Consideration be given to the extent to which it is feasible for all algebraic work to be learned within the context of vocational employment.*[13]
- *Explicit mention is made of the use of technology in connection with algebra.*

Within Engineering and Science Advanced GNVQs there is some algebra embedded into the mandatory units for these courses. This is not the case within what could be called 'Non-mathematical' GNVQs (e.g. 'Hospitality and Catering', 'Health and Social Care'). However, some GNVQs could be interpreted as being more mathematical than they currently are (e.g. Information Technology, Manufacturing, Management Studies, Construction and the Built Environment) with more mathematics embedded within the mandatory units.

Advanced GNVQ Science and Engineering clearly requires a great deal of algebra, extending well beyond that in 'Application of Number'. It seems unlikely that many students beginning Advanced GNVQ courses will have the algebra required, as many will have a modest grade from an intermediate GCSE course.[14] The MEI Foundation for Advanced Mathematics is a possible way forward here.

In GNVQ Science, there is no mandatory mathematics unit. This appears to be an anomaly. An analysis of some of the most elementary mathematical competencies required in the Science course has been carried out by the MAP project. Algebra is the key to most of these (see Appendix 4.2).

The mandatory unit of mathematics in GNVQ Engineering is a major leap forward from GCSE, taking the student as far as the beginning of calculus. However, mastery of differential calculus requires students, inter alia, to have some confidence and fluency with indices and literal algebra. The majority of Advanced GNVQ Engineering students are likely to have studied for the Intermediate Tier of GCSE mathematics and so will start the course with minimal experience of algebra. There is evidence that this unit has caused difficulties so far and these are not likely to be alleviated unless our recommendations for pre-16 algebra (Section 2) are implemented.

We recommend that:

- *Science GNVQ adopt as mandatory, an appropriate mathematics unit which includes significant algebra and function material (as for example, outlined in content Block A of Lord* et al. *(1995))*
- *Intermediate GNVQs are reviewed in order to examine the extent to which the marginalization of mathematics-algebra limits future progression both within GNVQ and within employment*

4.4 Access to higher education: the algebraic content of advanced GNVQ courses

There is no reason in principle why the needs of higher education should be considered when evaluating the mathematical-algebraic content of advanced GNVQs. However, in practice, universities are recruiting, or planning to recruit, students from GNVQ into a wide variety of courses. Moreover, students aspiring to enter higher education are being encouraged to see GNVQ as a possible route.

The minimum requirements of a number of courses were identified in Lord *et al.* (1995) and it is clear that algebra was a relevant element for many (see Appendix 4.2). In this report the comments of many admissions tutors emphasized confidence, feel for number and algebra, fluency and logical thinking. Such comments echo similar comments in 'Mathematics Matters in Engineering' (1995).

The significant lack of mathematics in general, and algebra in particular, within the bulk of Advanced GNVQ courses is likely to present considerable problems for students wishing to use this route as progression to higher education. Currently, students from a wide range of courses at higher education level follow a mathematics and/or quantitative methods course in their first year of higher education. 'Application of Number' is a totally inadequate preparation for such courses.

We recommend that:

QCA monitor the take up of HE places by GNVQ students and in particular the adequacy of the mathematics-algebra preparation of such students, with a view to recommending additional units within GNVQ Advanced level for a wider range of vocational areas.

[15] Changes (to the form of the GNVQ curriculum description) are currently being recommended.

4.5 Curriculum description: the development of algebraic thinking

An academic course is typically described in terms of a syllabus and assessment scheme or through expected learning outcomes and assessment scheme. In some cases a collection of curriculum tasks and materials may be included. The curriculum description within GNVQ extends this notion in great detail, identifying performance criteria, range within which the performance criteria are to be applied and a description of evidence indicators.[15]

Specifying and defining mathematical learning through competence statements has yet to be seen to be successful in encouraging the nature and approach to the learning of mathematics which is desirable. It can lead to a fragmentation which is at odds with the nature of mathematical knowledge which, as we have discussed throughout the report, is a complex web of interrelated ideas.

The external assessment of GNVQ has so far comprised multiple choice tests of the sort typical of 16+ courses of the 1970s. The internal assessments, when made, tend to checklist discrete items of mathematical knowledge and technique. This cannot be considered as serious evidence of mathematical achievement and competence. Mathematics and particularly algebra cannot be divided into fragmented facts.

The disadvantage of both this approach and the more 'traditional' form of curriculum description through a syllabus is that neither emphasizes explicitly the very aspects of algebraic activity which need to be encouraged.

Some recent advocates have argued that curriculum descriptions need to be supplemented with a description of the classroom processes which the teacher/lecturer will effect or encourage, and which should support the desired learning. This might be seen as equivalent to the existence of AT1 within the National Curriculum. However, there is a real danger here of emphasizing process in quite inappropriate ways, as has become the case with mathematical problem solving and investigations as discussed in Section 2.

An alternative approach needs to be found which makes sense to teachers and provokes teaching which makes it likely that students will learn mathematics which has some coherence and which becomes a valuable tool within the vocational context of mathematics. The principle here is to find a way of organizing the mathematics curriculum for vocational courses which relates to: what students need to learn, what they know when they enter such courses and the interrelated aspects of mathematics itself; teaching approaches which are effective (see Williams *et al.* (1996) for one such approach).
If the difficulties identified in the body of existing research on the learning of algebra (see Section 1.4) are to be tackled, the curriculum descriptions must be explicit about the key ideas and skills which students must tackle. These will

necessarily include statements about structure of arithmetic and hence algebra; statements about abstract representation, the precision and at the same time deliberate ambiguity of notation and symbols, in particular the changing use of the '=' sign from 'makes' to 'equals' as one moves from elementary arithmetic to more formalized arithmetic and algebra; the notion of multiple representations of the same idea; the key ideas of equivalence, inverse operation, variable and function; and the whole process of analytic thinking. At the same time they need to emphasize the 'feel for' and 'confidence with' aspects of learning algebra, suggesting that there needs to be much more explicit reference to the ability to talk about, discuss, argue and defend the particular uses made of algebra within vocational contexts.

Although the notion of competence and mastery may be useful for some aspects of algebra—for mastering specific skills and techniques—it is inadequate for such ideas as classification, comprehension of structure, condensation of ideas, representation and imagery.

Evidence from small scale studies (e.g. Johnson & Elliott, 1995) suggests that the emphasis on technical competence within a vocational context leads to low levels of self-confidence in algebra among students who have followed vocational courses in the 16–19 range compared with those who have followed an A-level route. This evidence also points to the prevalence among such students of an approach to algebra which Tall (1996) would call procedural.

4.6 Approaches to teaching: separate or embedded provision

The preceding sections indicate our concerns that the current model of GNVQ curriculum description is unhelpful in promoting adequate algebraic learning within vocational contexts.

Logically, this leads to the need to consider the form of curriculum provision in algebra, which for the most part is currently embedded within other vocationally oriented units. This may allow an emphasis on the adaptation and application of algebraic skills in context (setting up equations, considering the suitability of a proportionality model and interpretation of a solution). It does not allow a focus on the structure and sense of the algebra itself; these are intra-mathematical considerations which are the essence of algebraic development and which are currently inhibited by the demand for permeation through a vocational context. It should be recalled at this point that apart from those students following GNVQ engineering (who must complete a mathematics unit), the main mathematical-algebraic provision for all other GNVQ students lies in the core skill unit 'Application of Number', where evidence of performance is usually collected through vocational units. Some GNVQs incorporate additional mathematical-algebraic competencies within the performance criteria, but no separate units exist. (We have already discussed the particular problems this creates within science.)

Anecdotal evidence suggests that at Foundation level, 'Application of Number' tends to be delivered separately within basic numeracy units as many students have major problems with basic skills. Such courses are usually taught by numeracy staff within colleges, which itself creates sequencing and permeation difficulties.

At Intermediate level, there is often separate teaching for about one hour a week concentrating on arithmetic and data collection, again delivered by numeracy staff within colleges.

At Advanced level there is usually no separate teaching for the core skill application of number. Students learn their algebra through other units and have access to drop-in workshops within colleges, which utilize a mixture of paper-based and IT-based materials. The extent to which algebra at this level is developed appears to vary considerably, dependent in part on the time allocated to GNVQ and the particular assessment burdens upon individual staff and students. Evidence is accumulating (Molyneux & Sutherland, 1996) that this approach to teaching the Advanced level core skill makes it very unlikely that students will learn any algebra at all.

In contrast with schools, further education is not an all-graduate profession. It would appear that a considerable number of staff with low or in some cases no qualification in mathematics are being required to teach mathematics within the context of their own subject area. This could generate problems as these staff lack confidence in their own mathematical skills, and an understanding of algebra in particular, and thus appear hesitant to encourage students or to provide advice and support. Students experiencing difficulties are often referred to drop-in workshops which they may or may not attend.

On the other hand, if the vocational teacher does not have any involvement in the core unit then it can be ignored by them as it may be assumed to be delivered in basic mathematics units.

The permeation of 'Application of Number' across several other units presents its own problems of fragmentation. The coherence of the algebraic experience of students would appear to be almost impossible to plan, let alone achieve. There must be a case therefore for key ideas and skills to be developed through some specialist provision running alongside other units. This would help achieve coherent planning of the delivery of 'Application of Number' through those other units.
The assessment of 'Application of Number' is intended to be integrated within the assessment of the other units studied. There is some evidence to suggest that in some cases assessment of this core skill is in fact achieved using contextualized assignments set and marked by an 'Application of Number' specialist. Such staff are only in a position to identify evidence of achievement, and have no responsibility for delivery. It is the students' responsibility to

collect evidence of achievement, but again, there is evidence that many do not understand what is expected of them, so seek advice from the vocational unit staff, who feel unable to help and so pass them on. The separation of delivery and assessment created by the insecurity of staff themselves within the mathematical field, thus leads to poor practice and little gain in understanding or skill on the part of students.

The complexity of delivery of Application of Number' thus raises questions about the extent to which students are able to develop their own algebraic thinking through their course of study, and the extent to which the very fragmented nature of its delivery makes this impossible anyway.

More positively, within GNVQ, algebra is intended to be presented within a context, with all its potential implications of relevance, motivation, sense-making and rationale. However, we suggest that changes over the last ten years within academic education have not made algebra more accessible to the majority, but have more or less removed it from the curriculum.

We recommend that :

- *The teaching of the core skill unit 'Application of Number' be coordinated by a suitably qualified mathematics expert and delivered through a mixture of separate and embedded provision which enables both the appropriate contextualized application of algebraic competencies and the coherent development of algebraic thinking.*
- *QCA commission research into the extent to which:*

  'Application of Number' is being delivered currently by inappropriately qualified staff;

  GNVQ students are disadvantaged by the lack of confidence and ability of staff themselves to embed algebraic thinking and skills appropriately within vocational units and/or the implicit removal of the need for such staff to address such competence by the provision of entirely separate numeracy provision.
- *QCA consider, in expanding the notion of core skills to all students in the 16–19 age range, whether or not it is appropriate for all students to achieve the same level of knowledge and understanding of 'Application of Number'. Is the same core unit appropriate for all ?*

4.7 International comparisons

It has not been possible within the context of this report for us to make comparisons between the algebra components of vocational courses in France, Germany and England. However, we refer here to the work of Wolf (1996), who has pointed out that currently the approach to teaching and learning mathematics to vocational students in England and Wales is very different from other European countries such as France and Germany, where vocational courses contain mathematics units with a vocational emphasis, but which are taught separately. She states that 'Overall, therefore, mathematics teaching in English vocational courses is quite as unique as it is for academic ones, indeed perhaps more so, since in the former it is simply missing, while in the latter we find features and delivery patterns totally different from those used anywhere else. Particularly striking is the combination of a completely "common" list of skills, the same across all vocational areas, with an emphasis on completely integrated delivery. Also unusual is the absence of any guidance on time to be spent on teaching and learning mathematics, the uncoupling (in theory) of core skill and award levels, and the degree to which the design of teaching and assessment materials are delegated to the individual teacher or course team'. She also points out that 'The lack of ambition manifested in English vocational course design contrasts dramatically with the situation in Europe or the Pacific Rim, where objectives are set high with an express view of increasing national performance levels and ensuring that students are challenged and guaranteed the possibility of progression'(Wolf, 1996).

We recommend that:

- *QCA consider further the evidence from international competitors as to the mathematical-algebraic demands made through vocational courses. (See for example Wolf & Rapiau, 1993; Wolf, 1992; Wedege, 1995*)

4.8 Summary

For students to be successful with algebra at this level of education within the vocational field, there is a need for teachers to enable students to address past failure and to engage in wider conceptions of arithmetic and algebra than they may have to date.

The priority in teaching should therefore be to help students make sense of algebra (and pre-algebra) structurally, establishing connections with number, with geometry and graphs and between algebraic concepts such as equation, function and transformation. When a student has confidence in and a sense of the notion of procedures, then it will be profitable to practise specific techniques and become fluent with them.

Potential advantages of the approach to GNVQ delivery are the emphasis upon independent learning, resource-based study and

responsibility to provide evidence. There is a sense in which this ought to promote ownership by students and encourage deep approaches to learning (see for example Gibbs, 1992). However, the ability of students to embrace such approaches is limited by their perception of what constitutes algebra. Current curriculum descriptors and organization mitigate against a holistic conception, encouraging instead a piecemeal conception, focused around mastery of individual techniques. In addition vocational students need considerable teacher support to identify what they do and do not know about algebra before they can embark on independent learning. A denial of the role of the teacher again relates to a lack of understanding of the complexities of learning mathematics.

The group is somewhat sceptical of the ability of open access workshops and current computer-based learning material on their own, to alter students' prevailing conceptions of learning algebra. Furthermore, there is some evidence that those students most in need do not necessarily attend open learning workshops. What they seek is individual personal help, not materials to work with. Open access workshops may support superficial attempts to resolve difficulties by encouraging students to seek help with solutions to specific questions rather than encouraging them to sort out their difficulties on a wider front.

We recommend that research is conducted on :

- *Whether the predominant models of teaching basic numeracy (for example workshops, mathematics specific units) are effective.*
- *The nature and extent of the use of technology and how this relates to learning.*

5. Conclusions

This chapter starts with a brief account of the main ideas and their consequences in the form of some rather general proposals. Then the remaining sections, 5.4–5.11, outline in more detail the implications of these proposals.

5.1 Nature of school algebra

In our report we have identified three important components of school algebra namely:

- *Generational activities*—discovering algebraic expressions and equations;
- *Transformational rule-based activities*—manipulating and simplifying algebraic expressions, solving equations, studying equivalence and form;
- *Global, meta-level activities*—ideas of proof, mathematical structure, problem-solving. (This final component is not exclusive to algebra.)

We have also used the idea of 'symbol sense', whereby algebraic symbols are used not merely as formal and meaningless entities with which to juggle, but as powerful ways to solve and understand problems and to communicate about them.

Our overall conclusion is that, in England and Wales, an over-emphasis has been placed on generational activities and that the other aspects of algebra have received too little attention. Consequently symbol sense is not being properly established.

5.2 Algebra as a language

The algebraic language is required in order to develop awareness of mathematical objects and relationships, many of which are virtually impossible to manage otherwise. The rules and grammar of this language are needed in handling the transformational activities mentioned above. Without appropriate emphasis on the symbolic language such essential ideas as algebraic equivalence cannot be learned.

It has to be accepted that pupils will make mistakes with the algebraic language and that this is an inevitable part of learning a rule-bound system. Consequently pupils need extensive use and feedback on these mistakes before they reach post-16 and higher education.

Users of a language need to know when to use which aspect of a language within which problem situation. This has to be communicated by humans. Computer-aided instruction cannot teach students the subtleties of when to use a language.

5.3 Changing emphasis in school algebra

What constitutes school algebra in England and Wales has changed over the last 10 to 15 years and particularly with respect to schooling pre-16 which is what we discuss first. Influenced by the Cockcroft Report, a particular approach to school algebra has been taken which is characterized by: an emphasis on problem-solving related to real-world problems; an emphasis on relating algebra to pupils' informal methods; a de-emphasis on the role of symbols. Some undoubtedly good ideas have become institutionalized in our school curriculum in quite unintended and unpredicted ways. For example, activities such as generating expressions from patterns have been prioritized. Algebra is predominantly an activity which is internal to mathematics and which cannot be contorted into instant relevance; so it is not surprising that increased emphasis on realistic problem-solving has resulted in decreased emphasis on algebra. Furthermore, much of what is currently called school algebra is not algebra at all. Pattern-spotting as an isolated activity, and trial and improvement for solving a quadratic equation, are not algebraic activities, despite the current National Curriculum implying that they

are. The mere use of algebraic symbols does not imply algebraic activity. (This is probably the biggest confusion of all, and perhaps explains why solving equations using 'trial and improvement' is currently considered to be algebra.) **The reason we stress this is that if what is not algebra is called algebra then the whole community is seduced into believing that algebra *is* being taught.**

A multitude of factors, which include new forms of assessment, the introduction of the computer and an emphasis on making mathematics more relevant, have worked together to nudge out the transformational rule-based and global meta-level aspects of algebra from the school curriculum.

Algebra is not even explicitly mentioned in the GNVQ Application of Number Core Units (Levels 1, 2 and 3). The omission of algebra when number, shape, space and measure, and handling data are specified seems to be a manifestation of the desire to suppress algebra within the curriculum.

The National Curriculum is currently too unspecific and lacking in substance in relation to algebra. The algebra component needs to be expanded and elucidated—indeed rethought. Similar changes are required at other levels and in other units.

Note: For most of the committee it was unproblematic to draw attention to the fact that 'trial and improvement' methods of solving equations are not algebraic methods. This does not imply that 'trial and improvement' methods are not valuable; in fact, a whole branch of mathematics (numerical methods) centres around this idea. However, the issue has become very contentious, as if this attacks the very crux of what is considered to be important in school mathematics in England and Wales.

It is a widely-held view that trial and improvement is somehow natural and spontaneous. However, we believe that this method has become taught and examined to such an extent that for many pupils it is the 'official school method'. For example, 'trial and improvement' is often explicitly specified in GCSE examinations (as is the case in the example shown on page 8 of this report) and students will probably not receive any marks if they do not use this method. The converse does not appear to be the case, as most GCSE examination questions do not specifically ask pupils to solve a quadratic equation by an algebraic method.

We recommend that a study is carried out to examine:

(i) the extent to which pre-19 pupils make use of trial and improvement;

(ii) whether this provides a good introduction to algebraic methods or whether it makes it more difficult to learn algebra.

5.4 Implications for national curricula

There are many pressures which have worked together towards reducing mathematics in general, and algebra in particular, to senseless fragments. Such pressures associated with the National Curriculum include the introduction of Key Stages, levels of attainment and new forms of assessment. In addition, there is enormous pressure for schools to perform well in the league tables and national examinations, and so teachers are likely to teach 'just in time' for the test and not before. For example, the item 'Pupils justify their generalizations or solutions, showing some insight into the mathematical structure of the situation being investigated' is not specified until level 7 in the National Curriculum and so it is not likely to be taught until pupils are deemed to have reached this level. This leads not only to lack of challenge but also to quite inappropriate 'separating off' of a central aspect of algebra. In the case of the new vocational courses, the very nature of assessment by competence criteria fragments all mathematics, and in particular algebra.

Conceptions developed by students, which may have been functional in previous situations, can become obstacles to further learning. This idea that previous conceptions can become barriers to future learning was probably implicitly known by 'good' mathematics teachers. The current National Curriculum with its levels of attainment is likely to work against this type of teacher-knowledge, especially as an idea tested at a previous stage is then supposed to be understood for all time. In other words, the National Curriculum presents teachers with an inappropriate model for thinking about teaching and learning. In the case of algebra, this can result in teachers and pupils losing sight of its central role in mathematical activity and hence can work against the teaching and learning of algebra.

We recommend a reworking of algebra within the mathematics National Curriculum and a critical appraisal of the notion of levels on which this National Curriculum is predicated.

We recommend that the structuring of the National Curriculum should take into account both the need to preserve mathematical coherence and a consideration of how pupils learn algebra. We urge that the interrelationship between levels of attainment and Key Stage tests be re-examined.

We suggest that more research is needed to understand the relationship between what algebra is taught and what is learned. We recommend that in the A-level Common Core, content should not be separated from modes of algebraic activity.

We also recommend that there should be a coherent and clear curriculum for the mathematical element in vocational courses.

5.5 Implications for the timing of algebra teaching

We have identified a range of activities which we consider to be precursors to algebra. These include such activities as working with relationships, expressing relationships involving more than one operation and working with inverse arithmetic operations. *These should take place both in the primary and early secondary school.*

Algebra for citizenship is a crucial idea, as well as algebra for those who intend to follow more specialized courses in higher education. As discussed in Chapter 4, this should involve a feel for number structure and the pre-algebraic ideas of expressing and working with relationships, and focusing on language as a means of expressing relationships. It also involves the use of appropriate symbolism within specific contexts, including information technology. There is a strong case for introducing algebra to all pupils, as in many other countries such as France and Germany. **We recommend that algebra is introduced from the beginning of secondary school with more emphasis being placed on all aspects of algebra.**

Nowadays many students have to devote much valuable time to the development of algebraic ideas at the start of any A-level mathematics course, and time throughout that course has to be found for students to continuously work on these algebraic activities. **We recommend that post-16 institutions develop bridging algebra courses for some students before they start A-level.**

Students following vocational courses are also often handicapped by a lack of experience of algebra and this is particularly the case in GNVQ science and engineering. **We recommend that QCA monitor the take up of higher education places by GNVQ students and in particular the mathematics-algebra preparation of such students, with a view to recommending additional units within GNVQ advanced level for a wider range of vocational areas.**

5.6 Implications for teaching algebra

To be effective a teacher has to become aware of pupils' individual approaches but also has to orchestrate learning so that pupils develop knowledge of mathematics which is recognized by communities outside school. In the case of algebra this involves the teacher more or less imposing an algebraic language which pupils will not previously have encountered.

Algebraic problems within schools will always have to be contrived when relating to the 'real' world. Algebra word problems were traditionally used for this purpose. In England and Wales these types of problem have mostly been rejected as being too contrived, although Bell (1995) has shown how they can be used in a creative way in the classroom. The current trend is to introduce algebra within supposedly realistic contexts which centre around patterns of objects (for example chairs and tables, tiles around a pond, matchsticks) and which are no more realistic than the algebra word problems which were used in the past. Within these types of problems, the introduction of the algebraic modes of thinking and analysing tends to be delayed and is not viewed as central to the problem-solving process. Problem-solving related to 'realistic' situations and mathematical modelling often depends on algebra but does not necessarily make a good vehicle for teaching algebra. *Problem situations have to be devised in which it makes sense to introduce algebraic concepts and in which teachers are not fearful to talk about something which pupils cannot yet know about.*

Algebra has to be taught and the teacher will always play a crucial role in this respect. In order to do this, teachers have to understand what school algebra is. If teachers take a routine approach to what is currently specified as algebra within GCSE examinations, Key Stage tests, curriculum materials and the National Curriculum, then it is not likely that algebra will be taught. **This is the reason why teachers cannot be blamed for the current situation.** This is also the reason why we are calling for changes in the curriculum and its assessment. Algebra is perhaps the central part of mathematics, and *it should be taught as such.* (We are not arguing for separate algebra courses as has been the case in the USA, for example; but we are arguing for algebra to be

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LIST OF APPENDICES

Appendix 1.1 A-level Points: 10th/90th Percentile For 1994 Entrants Studying Mathematics (G100)/Physics (F300)*

	Maths	Physics
Bangor	8/24	-
Bath	16/30	10/24
Birmingham	20/30	14/30
Bradford	6/20	-
Bristol	18/30	14/30
Cambridge	-	-
Cardiff	8/22	12/30
UC Lancs	6/14	-
City	10/20	-
de Mont.	6/18	-
Durham	20/30	16/30
Edinburgh	22/30	18/30
Essex	10/23	14/20
Glasgow	18/22	-
Heriot-Watt	18/30	-
Hull	8/24	-
Imperial	18/30	16/30
Kent	14/16	12/24
Kings College	14/28	14/30
Lancaster	22/28	6/18
Leeds	14/30	18/26
Leicester	10/26	14/20
Liverpool	10/30	8/24
Loughborough	14/26	16/24
Manchester	16/28	16/30
UMIST	14/26	20/30
Newcastle	16/28	12/30
Nottingham	20/30	20/30
Oxford	24/30	28/30
Reading	18/23	10/30
RH, U of London	16/28	10/16
Sheffield	16/28	14/30
Sheffield Hallam	8/14	4/13
Southampton	10/28	10/24
St. Andrews	26/30	18/20
Surrey	14/24	10/24
Sussex	-	-
Swansea	12/24	10/24
UCL	16/30	20/26
Warwick	20/30	16/28
York	14/28	14/30

NOTE 1: Figures based on students accepted pre-clearing and A-levels taken in the year of entry and the year prior to entry to HE. However, they also include mature students who may have taken only 1 A-level.

NOTE 2: Missing figures may mean no course, or no figure available to UCAS.

* Source UCAS 1996 Official Guide.

Appendix 1.2 School algebra in Germany and France

GERMANY - Overview of School System

Schools in Germany are organised by the *Länder*. Although there are differences in the detail, for example the school-leaving exams can be set either by the *Land* or by individual schools, they all follow more or less the same pattern. Children start formal school at the beginning of the academic year in which they are seven. After four years in the *Grundschule* students attend one of three types of school (the decision about which school to attend is made either on the basis of an 11-plus style examination or by the *Grundschule* depending on the *Land*):

Hauptschule 5 further years with 4 hours a week of mathematics. The school-leaving qualification is the *Hauptschulabschluss*.

Realschule 6 further years with 4 hours a week mathematics. The school-leaving qualification is the *Realschulabschluss.*

Gymnasium 9 further years with 4 hours a week mathematics during the first 7 years. For the final two years at the Gymnasium, students must decide to take either the:

Mathematik-Grundkurs (basic course) with 3 hours a week mathematics, or the

Mathematik-Leistungskurs (advanced course) with 5 hours a week mathematics.

It is not possible to drop mathematics. Between 25% and 30% of *Abitur* students take the (more advanced) Leistungskurs. Students usually take 6 *Grundkurse* and 2 *Leistungskurse.*

The school-leaving qualification is called the *Abitur*. This is graded on the basis of assessed work in all the subjects throughout the final two years at school and on the basis of final written and oral examinations. The final written examinations are for the two *Leistungskurse* and one of the *Grundkurse*. Students are examined orally in one further *Grundkurs*.

The national figures show that around 31% of children attend the Gymnasium, 26% the Realschule and 26% the Hauptschule. (There are some comprehensive schools in some *Länder*. These take only a very small proportion (9%) nationally of any age-group.[16] There are also some combined *Haupt-* and *Realschulen* which take around 7.5% of children) Figures for Baden-Württemberg suggest that around 27% of any age-group obtain the *Abitur*.
It is possible (and not uncommon) for students to drop down from the *Gymnasium* to the *Realschule* if they have trouble coping. It is also normal for students to have to repeat a year if they fail a year's work. This means that, although the 'normal' age for leaving the *Realschule* and *Gymnasium* is 16/17 and 19/20 respectively, students can be significantly older.

FRANCE - Overview of school system[17]

Schools in France are organised on the basis of national programmes under the responsibility of the Minister of National Education. In France pupils start school at the beginning of the school year in which they are six. After five years in the École Primaire all pupils enter the Collège (which is a comprehensive system). The École Primaire and Collège constitute compulsory schooling. After Collège students study at the Lycée. The following is a breakdown of the mathematics studied at the Collège and the Lycée.

Collège 4 years

During the first two years of collège all pupils study the same mathematics (with no streaming or setting). Each pupil has between 3–4 hours of mathematics a week (the choice is made by each collège).

During the last two years of collège there are two possible courses:

- a general route
- a technological route

The examination at the end of Collège is le *Brevet des Collèges.* The subjects are planned at the level of the Academy (Academies are organised regionally). This examination has no influence on the passage to higher levels of study.

Lycée 3 years

At the Lycée there are three possible courses:

- a general route
- a technological route
- a vocational route

General and technological route. The first year of these two routes is common i.e la classe de second, with 3 and a half hours of mathematics plus about one hour of 'group work' (modules).

Students are prepared for the Baccalaureat in the last two

[16] Structures of the Educational and Initial Training Systems in the EU European Commission, Official Publication of DGXXII, 1995.
[17] We are indebted to Martine Desigaux, Collège Jules Flandrin, Corenc, Isère, France for providing this information.

years, première and terminale.
The last two years of the *General Baccalaureat* prepare for the Baccalaureat in three courses, the options being different in each:

> Bac L—literaire, with no more compulsory mathematics but the possibility of taking an optional mathematics course (4 hours);
>
> Bac ES—economique et social, with a compulsory mathematics course applied to economics and social sciences (3 hours in première and 4 hours in terminale), with a possibility of two optional supplementary hours in the première and the terminale class.
>
> Bac S—scientifique, with compulsory mathematics: 5 hours in première, 6 hours in terminale. In addition, in première a supplementary hour in groups and in terminale an optional two hours in groups.

The last two years of the *Technological Baccalaureat* consists of 8 principal possible orientations with differing hours of mathematics (in the following hours for première are followed by hours for terminale).

> Science et technologie industrielle (3 plus 1 hour of 'group' module, then 4 hours); Science et technologie de laboratoire (3 or 4 plus 1 hour of module, then 2 or 4 hours); Sciences medico-sociales (3 plus 2 hours); Science et technologie tertiare (2-3 hours, with the possibility of 1 hour more in a module, the same in première and terminale); Science et technologie du produit agro-alimentaire (at least 3 plus 3 hours); Hotellerie (2 plus 2 hours); Arts appliqués (3 plus 3 hours); Techniques de la musique et de la danse (4 plus 3 hours).

The *vocational route* has many possible strands.

> In order to obtain a Brevet d' Etudes Professionnelle (BEP) in 2 years there are two options: either industrial route with 4 hours of mathematics and physics together in the 1st and 2nd year, or tertiary route with 3 plus 2 hours of applied mathematics. After the BEP students can eventually rejoin the Bac professionnel which corresponds to a specific cycle (not to be confused with the voie technologique). One in two students follow this direction after the BEP.

Pupils can repeat a year of schooling in France, average rate of repeating: Terminale—17%; Premiére—8%; Seconde—17%; Troisième—10%; Quatrième—7.3%; Cinquième—11.2%; Sixième—10.1%

School Algebra in France and Germany

Primary School. Analysis of the 1995 test used with pupils in the last class of the École Primaire suggests that pupils are not given so much intermediate support to answer 'arithmetic type' questions as is the case with Key Stage 2. The French test places more emphasis on choice of operations than process of computation. No similar comparison can be made with the German system as pupils are not tested at this stage in Germany.

Lower Secondary School. The Brevet examination is passed by approximately 75% of an age cohort in France. Calculators are allowed in the examination. However the process of using a calculator is explicitly examined in GCSE, which is not the case in the Brevet. Despite the fact that calculators are allowed in the Brevet examination, pupils are sometimes explicitly asked to present their answers in a particular form which could not be produced by a calculator (e.g. leave in square root form). In this sense the Brevet questions place more emphasis on structure and form than GCSE questions. Brevet examination questions are rarely situated in practical contexts. Diagrams are presented as analytical tools. There is no 'pictorial' decoration. Brevet questions examine a much narrower range of mathematical ideas than the GCSE examination.

Algebra questions in the Brevet involve very explicit manipulation of symbols. Brevet questions do not explicitly test 'trial and improvement' methods. The algebra and arithmetic Programme specified for all pupils in the Classe de Troisième is shown below. This shows the considerable emphasis which is placed on structure in arithmetic together with algebraic equivalences.

2. TRAVAUX NUMÉRIQUES

La *résolution de problèmes* (issus de la géométrie, de la gestion de données, des autres disciplines, de la vie courante) constitue l'objectif fondamental de cette partie du programme.

La pratique du calcul exact et approché doit conduire, à l'issue de la classe de Troisième, à une bonne maîtrise des règles opératoires et des règles de comparaison des nombres.

L'entraînement au calcul littéral se poursuit et doit aboutir à une relative autonomie.

1. *Écritures littérales; factorisation d'expressions de la forme:*

$a^2 - b^2$, $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$

Comme en Quatrième, les travaux s'articuleront suivant deux axes:

Utilisation d'expressions littérales pour des calculs numériques;

Utilisation du calcul littéral dans la mise en équation et la résolution de problèmes divers.

Les activités viseront à assurer la maîtrise du développement d'expressions simples; par contre, la maîtrise de la factorisation n'est pas un objectif de la classe de Troisième. On entretiendra les compétences en matière de calcul sur les puissances.

Savoir factoriser des expressions, telles que:

$(x + 1)(x + 2) - 5(x + 2)$

$(2x + 1)^2 + (2x + 1)(x + 3)$

Connaître les égalités

$(a + b)(a - b) = a^2 - b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

et savoir les utiliser sur des expressions numériques ou littérales simples, telles que:

$101^2 = (100 + 1)^2 = 100^2 + 200 + 1$

$(x + 5)^2 - 4 = (x + 5)^2 - 2^2$

$= (x + 5 + 2)(x + 5 - 2)$

2. *Calculs élémentaires sur les radicaux (racines carrées):*
Produit et quotient de deux radicaux.
Puissance d'ordre 2 ou 4 d'un radical.

La touche $\boxed{\sqrt{\ }}$ de la calculatrice, qui a déjà été utilisée en Quatrième, fournit une valeur approchée d'une racine carrée. On met en place, par ailleurs, les règles de calcul ci-contre.

Le calcul sur des expressions comportant des radicaux (telles que $\sqrt{3 + 2\sqrt{2}}$, $\frac{1}{\sqrt{2} + 1}$) n'est pas un objectif du programme. Comme dans les classes antérieures, on habituera les élèves à écrire un nombre sous la forme la mieux adaptée au problème posé.

Savoir que, si a désigne un nombre positif, $\sqrt{a}$ est le nombre positif dont le carré est a.

Savoir déterminer, sur des exemples numériques, les nombres x, tels que $x^2 = a$, où a désigne un nombre positif.

Sur des exemples numériques, utiliser les égalités:

$(\sqrt{a})^2 = a$, $\sqrt{a^2} = a$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, où a et b désignent deux nombres positifs.

Par exemple: $\sqrt{45} = 3\sqrt{5}$,

$\sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$, $\frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$.

There have been a number of studies of the three German school-leaving examinations with their English counterparts (Chandler, 1996,[18] Steedman, 1997[19])

Hauptschule/Realschule.
The *Hauptschulabschluss* is designed for the least academically-minded students and is the school-leaving qualification (taken at 15/16) of about 30% of the population. The comparable qualification would therefore be GCSE Basic Tier. In Bavaria, only 5% of school-leavers fail to obtain the competence required of them, whereas 15% of 16-year-olds in England and Wales fail to gain a GCSE certificate in mathematics

The *Realschulabschluss* is the school-leaving qualification (taken at 16/17) obtained by about 35% of the population. According to figures quoted in Steedman (1997) around 65% of all German students obtain the corresponding level in mathematics (this includes those in the *Realschulen* and those in the *Gymnasiun*). Around 43.5% of English and Welsh students obtain five GCSEs at Grades A–C.[20]

Chandler (1996) compared questions from Bavarian Hauptschule examinations with those from GCSE basic tier, and compared questions from the Realschule examinations with GCSE higher level. She found that much lower expectations were placed on English and Welsh students than on their Bavarian counterparts. The main difference is in the way questions in GCSE are broken down so that English and Welsh students are required to show 'considerably less ability to reason through a problem than their counterparts in Bavaria'. She suggests that 16-year olds in Bavaria, at the equivalent of the GCSE tier are expected to handle complicated algebraic expressions, many of which are not found until A-level in the UK.

Upper Secondary School.
We have studied questions from the *Abitur* examinations from the last six years, including the attached sample questions. Although the questions probably appear very difficult to UK based students and teachers, this may in part be due to differences in approach. There appears to be much less emphasis on the relevance of mathematics and much more on geometry and the manipulation of abstract concepts. The structure of the question is almost identical every year, so that it is possible for students to practise extensively on questions from previous years. In relation to algebra we list our key conclusions below. However we would encourage anyone interested to study the questions for themselves. (We have appended them to this section for that purpose).

1. The *Grundkurs* requires comparable algebraic manipulative skills to those required in single mathematics A-level.

2. The *Grundkurs* requires much greater skill in formulating problems algebraically than does either single or further mathematics A-level.

3. The *Grundkurs* is compulsory for all students at the *Gymnasium* not just those specialising in mathematics (although not all of these students are assessed on the basis of written examination). Teaching all students to something like this level in the UK would obviously require a larger number of well-qualified and able teachers than are available today.

4. The *Leistungskurs* requires a comparable or even higher level of algebraic skills to those required in Further Mathematics A-level.

5. The *Leistungskurs* also requires much greater skill in formulating problems algebraically before attempting to solve them than does further mathematics A-level.

We have not made such an extensive comparison with the Baccaluareat examination in France. We include an example on complex numbers from the Bac S. This gives a sense of the algebraic competence expected of students.

5 SUJET COMPLÉMENTAIRE
JUIN 1995, SÉRIE S ✻ *5 points*

On note i le complexe de module 1 et d'argument $\frac{\pi}{2}$.

Soit $P(z) = z^3 - 8z^2 + 24z - 32$, où z est un nombre complexe.

1. *a*) Vérifier que $P(4) = 0$.

b) Déterminer les nombres réels a, b et c tels que, pour tout nombre complexe z, $P(z) = (z-4)(az^2 + bz + c)$.

c) Résoudre dans l'ensemble des nombres complexes l'équation $z^2 - 4z + 8 = 0$ (on donnera la forme algébrique des solutions). En déduire les solutions de l'équation $P(z) = 0$.

2. Le plan complexe est muni d'un repère orthonormal $(O; \vec{u}, \vec{v})$ d'unité graphique 1 cm. On note A, B, C les points d'affixes respectives : $2 + 2i$; 4 ; $2 - 2i$.

a) Placer ces trois points.

b) Montrer que OA = OC = AB = CB.

c) Donner la forme trigonométrique de $z_A = 2 + 2i$ et de $z_C = 2 - 2i$.

d) En déduire la nature du quadrilatère OABC.

[18] Chandler, 'A comparison of Public Expectation of Achievement in Mathematics in England and Wales and Bavaria—A Personal View'. Report to SCAA, 3/7/96.

[19] Steedman H., Greer A., Bertrand O., Richter A, Robin M. & Loeber K., (1997) Assessment Qualifications and Standards: the UK compared to France, Germany, Singapore and the US, CEP Skills Audit, DFEE

[20] National School League Tables, 21/11/95.

Sample Examination Questions
Baden-Württemberg 1995

tudents taking the *Grundkurs* would have three hours mathematics tuition a week in their final wo years at the *Gymnasium*. Students specialising in mathematics take the *Leistungskurs* which ypically involves 5 hours a week during the final two years.
S

he examination for both the *Grund-* and *Leistungskurs* consists of a four hour written xamination involving two questions. There is no choice of question—the teacher selects the uestions from a choice of three in each of the relevant areas. Students must study analysis and ne of linear algebra or probability.
T
e

NALYSIS—Grundkurs
function f is defined for every t by

$$f_t(x) = e^{2t-x} + x - 3t; \qquad x \in \mathbb{R}.$$

A
ts graph is K_t.

a) Investigate K_1 for extrema, turning points and asymptotes.
Plot K_1 and its asymptotes for $0 \leq x \leq 5$. (Use 1cm = 1 unit.)
Calculate the area enclosed by K_1 and the coordinate axes.

b) The tangent to K_1 at $P(u|f_1(u))$ with $0 \leq u \leq 2$ cuts the y–axis at Q.
The straight lines $x = -1$, $x = u$, $y = -3$ and the line parallel to the x-axis through Q define a rectangle. This rectangle has area $A(u)$.
Calculate $A(u)$, $A(0)$ and $A(2)$.
What is the maximum value of $A(u)$?

c) Calculate for general t the coordinates of the minimum T_t of the curve K_t.
Show that all points T_t are equidistant from the straight line $y = -\frac{1}{2}x$.
For which values of t does T_t lie on one of the coordinate axes?

d) Consider the region which stretches to infinity to the right of the y–axis and which lies between each curve K_t and its asymptotes.
Prove that its area J_t is finite for all t.
What is the relation between t_1 and t_2, if $J(t_1) = eJ(t_2)$?

EOMETRY AND LINEAR ALGEBRA—Grundkurs
onsider the point $Q(1|1|-3)$ and straight lines

$$f : \vec{x} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad g : \vec{x} = \begin{pmatrix} 4 \\ 4 \\ -3 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \qquad r, s \in \mathbb{R}.$$

a) Find the coordinates of the point of intersection S of the straight lines f and g.
Calculate the angle of intersection of the straight lines.
The plane E contains f and g.
Derive the equation of the plane E.
Calculate the distance between the point Q and E.
The straight line h passes through the point S and is perpendicular to the both f and g.
Find an equation for h.

b) The sphere K passes through the origin O, touches the plane E from part a) and has the smallest possible radius.
Find an equation defining K.
The sphere K^* is the mirror-image of K taken with respect to E. Find an equation for K^*.
What are the coordinates of the point B, at which the two spheres touch?
Justify the statement, that the straight line through the centers of the spheres is parallel to the straight line h from part a).
Calculate the distance of the straight line h from the sphere K.

c) The straight line p is parallel to the straight line g and passes through the point Q.
A set of spheres includes all spheres which have centres on p and which touch the plane E.
Derive an equation for the spheres in the set.

Investigate whether the sphere K from part b) is a member of the set.
The points of contact with the plane E of all the spheres in the set lie on a straight line.
Give an equation for this straight line

ANALYSIS—Leistungskurs
For every $a > 0$ the function f_a is given by

$$f_a(x) = \frac{\ln(ax)}{x^2}; \quad x > 0.$$

Its graph is K_a.

a) Investigate K_a for maxima, minima and asymptotes.
Find the equation of the line C on which all maxima of K_a lie.
Plot K_4 and C for $0 < x \leq 3$. (Use 2.5cm = 1 unit.)

b) Find the equation of the tangent to K_a, which passes through the point $P(u|f_a(u))$.
How should u be chosen so that this tangent passes through the origin O?
For which a does this tangent through the origin bisect the angle between the x- and y-axes.

c) The function f_a has an integral of the form

$$F_a(x) = \frac{D + E \ln x}{x}$$

Find the coefficients D and E.
The curve K_a, the x-axis and the straight line $x = t$ with $t > \frac{1}{a}$ define a surface with area $A_a(t)$.
Find $A_a(t)$ and the image of the function $t \mapsto A_a(t); \quad t > \frac{1}{a}$.

d) Prove that for all $x > 0$ the inequality $\ln x < \sqrt{x}$ holds.

GEOMETRY AND LINEAR ALGEBRA—Leistungskurs
Consider a cartesian coordinate system and the points $F(2|0)$, $G(6|4)$, $A(-2|2)$ and $A'(6|-2)$. An affine transformation α has the fixed points F and G and maps the point A onto the point A'.

a) Construct the image of the straight line $h : 3x_1 - 2x_2 + 10 = 0$ and the pre-image of the straight line $g' : x_2 = -2$. (Place the origin at the centre of the graph paper and use 1cm = 1 unit.)
Describe the arguments you use in your construction.
Find the equation of the transformation α.
Specify the eigen values and eigen vectors as well as the fixed elements of α.
What kind of transformation is α?

b) The circle K with centre at A passes through the origin O.
It is mapped by the affine transformation onto the ellipse E.
In a new diagram demonstrate a construction to find the apices of this ellipse.
Calculate the area of the ellipse.
The vectors

$$\begin{pmatrix} u \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ u \end{pmatrix}, \quad u \in \mathbb{R},$$

are orthogonal.
Find u, so that the images of these two vectors under the transformation α are also orthogonal.

c) A dilation β with origin at $F(2|0)$ maps the point $P(3|0)$ onto the point $\bar{P}(4|0)$.
Give the equation of the mapping β.
Define a new origin and basis vectors, so that both α and β have the simplest possible representation.
Give the equations of the two transformations in this new coordinate system.
Demonstrate that $\alpha \circ \beta$ and $\beta \circ \alpha$ give the same transformation.

Appendix 2.1 1995 National Curriculum Programme of Study

Attainment Target 2: Number and Algebra

■ Level 1

Pupils count, order, add and subtract numbers when solving problems involving up to 10 objects. They read and write the numbers involved. Pupils recognise and make repeating patterns, counting the number of each object in each repeat.

■ Level 2

Pupils count sets of objects reliably, and use mental recall of addition and subtraction facts to 10. They have begun to understand the place value of each digit in a number and use this to order numbers up to 100. They choose the appropriate operation when solving addition and subtraction problems. They identify and use halves and quarters, such as half of a rectangle or a quarter of eight objects. They recognise sequences of numbers, including odd and even numbers.

■ Level 3

Pupils show understanding of place value in numbers up to 1000 and use this to make approximations. They have begun to use decimal notation and to recognise negative numbers, in contexts such as money, temperature and calculator displays. Pupils use mental recall of addition and subtraction facts to 20 in solving problems involving larger numbers. They use mental recall of the 2, 5 and 10 multiplication tables, and others up to 5×5, in solving whole-number problems involving multiplication or division, including those that give rise to remainders. Pupils use calculator methods where numbers include several digits. They have begun to develop mental strategies, and use them to find methods for adding and subtracting numbers with at least two digits.

■ Level 4

Pupils use their understanding of place value to multiply and divide whole numbers by 10 or 100. In solving number problems, pupils use a range of mental and written methods of computation with the four operations, including mental recall of multiplication facts up to 10×10. They add and subtract decimals to two places. In solving problems with or without a calculator, pupils check the reasonableness of their results by reference to their knowledge of the context or to the size of the numbers. They recognise approximate proportions of a whole and use simple fractions and percentages to describe these. Pupils explore and describe number patterns, and relationships including multiple, factor and square. They have begun to use simple formulae expressed in words. Pupils use and interpret co-ordinates in the first quadrant.

■ Level 5

Pupils use their understanding of place value to multiply and divide whole numbers and decimals by 10, 100 and 1000. They order, add and subtract negative numbers in context. They use all four operations with decimals to two places. They calculate fractional or percentage parts of quantities and measurements, using a calculator where appropriate. Pupils understand and use an appropriate non-calculator method for solving problems that involve multiplying and dividing any three-digit by any two-digit number. They check their solutions by applying inverse operations or estimating using approximations. They construct, express in symbolic form, and use simple formulae involving one or two operations.

Appendix 2.1 continued

■ Level 6

Pupils order and approximate decimals when solving numerical problems and equations such as $x^2 = 20$, using trial-and-improvement methods. Pupils are aware of which number to consider as 100 per cent, or a whole, in problems involving comparisons, and use this to evaluate one number as a fraction or percentage of another. They understand and use the equivalences between fractions, decimals and percentages, and calculate using ratios in appropriate situations. When exploring number patterns, pupils find and describe in words the rule for the next term or nth term of a sequence where the rule is linear. They formulate and solve linear equations with whole number coefficients. They represent mappings expressed algebraically, interpreting general features and using graphical representation in four quadrants where appropriate.

■ Level 7

In making estimates, pupils round to one significant figure and multiply and divide mentally. They understand the effects of multiplying and dividing by numbers between 0 and 1. Pupils solve numerical problems involving multiplication and division with numbers of any size, using a calculator efficiently and appropriately. They understand and use proportional changes. Pupils find and describe in symbols the next term or nth term of a sequence where the rule is quadratic. Pupils use algebraic and graphical methods to solve simultaneous linear equations in two variables. They solve simple inequalities.

■ Level 8

Pupils solve problems involving calculating with powers, roots and numbers expressed in standard form, checking for correct order of magnitude. They choose to use fractions or percentages to solve problems involving repeated proportional changes or the calculation of the original quantity given the result of a proportional change. They evaluate algebraic formulae, substituting fractions, decimals and negative numbers. They calculate one variable, given the others, in formulae such as $V = \pi r^2 h$. Pupils manipulate algebraic formulae, equations and expressions, finding common factors and multiplying two linear expressions. They solve inequalities in two variables. Pupils sketch and interpret graphs of linear, quadratic, cubic and reciprocal functions, and graphs that model real situations.

■ Exceptional performance

Pupils understand and use rational and irrational numbers. They determine the bounds of intervals. Pupils understand and use direct and inverse proportion. In simplifying algebraic expressions, they use rules of indices for negative and fractional values. In finding formulae that approximately connect data, pupils express general laws in symbolic form. They solve problems using intersections and gradients of graphs.

Appendix 2.2 Example of use of spurious context for teaching algebra

A Expressions

The manager of a supermarket has to check the stock at the end of each week.

A1 At the beginning of a week there were 740 cans of Fizi-Cola in the supermarket.

During the week 230 cans were delivered to the supermarket, and 410 cans were sold.

How many cans were there in the supermarket at the end of the week?

A2 At the start of a week in a hot summer there were 3800 cans in stock.
1200 were delivered during the week and 1700 were sold.

How many cans were there at the end of the week?

Every week the manager has to do the same kind of calculation.

He starts with the number at the beginning of the week.

He adds on the number delivered.

He subtracts the number sold.

The numbers change from week to week. They are **variable**. But he does the same kind of calculation with them every time.

We can use letters to stand for numbers which vary.

Let b stand for the number at the beginning of a week.
Let d stand for the number delivered to the supermarket.
Let s stand for the number sold.

The manager does the calculation $b + d - s$.

A calculation with letters standing for numbers is called an **expression**. The expression $b + d - s$ is the number of cans in stock at the end of a week.

A3 (a) Does the manager get the same result if he
starts with the number at the beginning of a week.
then subtracts the number sold, then adds the number delivered?
(b) Write the expresseion for this.

Expressions

A4 Which of these expressions also give the number at the end of a week?

(a) d - s ÷ b (b) b - s ÷ d (c) d - s - b
(d) d - b - s (e) s - d - b (f) b ÷ s - d

EQUIVALENT EXPRESSIONS

The expressions and
always give the same result.
no matter what numbers b.d.s stand for.
They are called equivalent expressions.

A5 Write down as many other expressions as you can which are equivalent to b ÷ d - s.

A6 Rajesh is given some money at the beginning of the day.
He spends some on sweets, some on comics, and some on bus fares.

Let g stand for the amount he is given, in pence.
Let s stand for the amount he spends on sweets.
Let c stand for the amount he spends on comics.
Let f stand for the amount he spends on fares.

(a) Write down an expression for the number of pence he has left.
(b) Now write down an many other expressions as you can which are equivalent to the first one.

Appendix 2.3 Example of pupil's work in which algebraic symbols are used to generate relationships

Generalising and formulating patterns and constraints, reasoning in generalised arithmetic

A rich activity that encourages students to use and develop understanding of symbols for generalised arithmetic involves looking for patterns in "windows" on different number grids. This task can be done as a pre-algebra activity with younger students. There is plenty of opportunity for older students to apply and refine their algebraic skills. One Y 11 students was looking at 3 by 3 windows on a hundred square.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
2								29	30
3								39	40
4			44	45	46			49	50
5			54	55	56			59	60
6			64	65	66			69	70
7								79	80
8								89	90
91	92	93	94	95	96	97	98	99	100

Adding opposite corners

= 44 + 66 = 110

46 + 64 = 110.

54 + 56 = 110

45 + 65 = 110.

All four results are the same.

She observed that her result about opposite corners appeared to be true wherever she positioned the window. She was able to prove this result using algebra.

Proof :-

a	$a+1$	$a+2$
$a+10$	$a+11$	$a+12$
$a+20$	$a+21$	$a+22$.

1) $a + a + 22 = 2a + 22$
2) $a + 2 + a + 20 = 2a + 22$
3) $a + 1 + a + 21 = 2a + 22$
4) $a + 10 + a + 12 = 2a + 22$

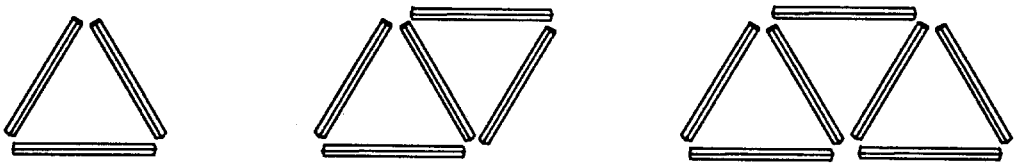
} the same.

By substituting in a number for 'a' we can see that the equation above would work for any number on the number grid :-

Appendix 2.4 **Key Stage 3 – Algebra assessment item**

9.

These patterns are made with matchsticks.



1 triangle
3 matchsticks

2 triangles
5 matchsticks

3 triangles
7 matchsticks

Every pattern is made with an odd number of matchsticks.
The rule for finding the number of **matchsticks** in a pattern is:

> **2 times** the number of triangles,
> **add 1.**

(a) Jason wants to make the pattern with **9** triangles.
Use the rule to find how many matchsticks he will need.

.....matchsticks

....
1 mark
3/4b

(b) **M** = number of **matchsticks**
T = number of **triangles**

Use symbols to write down the rule connecting **M** and **T**.

Appendix 2.4 Continued

(c) The rule for finding the number of **triangles** in a pattern is:

> The number of matchsticks **take away 1**, then **divide by 2**.

Bethan uses **11** matchsticks to make a pattern.

Use the rule to find how many triangles she has in her pattern.

.....triangles

1 mark
3/4b

(d) Misa uses **35** matchsticks to make a pattern.

Use the rule to find how many triangles she has in her pattern.

.....triangles

1 mark
3/4b

Appendix 3.1 SMP 16–19 Mathematics A-level 1995

Section A **Answer all questions.**

1 (a) Write $x^2 + 8x + 5$ in completed square form. *(3)*

(b) Find the coordinates of the minimum point of the graph of

$$y = x^2 + 8x + 5.$$ *(1)*

(c) Hence write down the coordinates of the minimum point of

$$y = (x + 1.2)^2 + 8(x + 1.2).$$ *(2)*

(6)

2 One particular LP record lasts for 24 minutes when played at a speed of $33\frac{1}{3}$ revolutions per minute.

(a) Find the number, n, of revolutions the record makes when it is played. *(1)*

(b) The groove on the record can be modelled by assuming that it is n concentric circles of regularly decreasing circumference. The diameter of the outermost circle is 29 cm and that of the innermost circle is 13 cm. Calculate the total length of the groove using this model. *(4)*

(5)

3 The value, £V, of a particular motorcycle t years after 1990 can be modelled by

$$V = 3500e^{-0.15t}$$

(a) Write down the value of the motorcycle when new in 1990, and obtain an estimate for its value 5 years later. *(3)*

(b) Find the value of $\frac{dV}{dt}$ when $t = 7$, and describe clearly what this number represents. *(5)*

(8)

4 The weight of an average male child, w kg, at age m months can be modelled using the equation

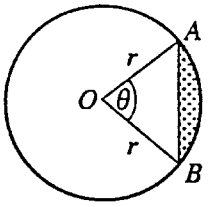
$$w = 3.5\ln(m + 2) + 1, \qquad \{m \in \mathbb{R} : 0 \leqslant m \leqslant 24\}.$$

Calculate the likely age of a male child of weight 10 kg and find the rate at which the child is gaining weight at this time. *(6)*

(6)

Appendix 3.1 continued

5 The diagram shows a circle, centre O and radius r. Angle $AOB = \theta$ radians.



The area of the triangle OAB is $\frac{1}{2}r^2 \sin\theta$.

(a) Explain why the area of the shaded segment is

$$\frac{1}{2}r^2(\theta - \sin\theta).$$ *(2)*

(b) When the area of the triangle OAB is twice the area of the shaded segment, show that the value of θ is given by

$$\theta = 1.5 \sin\theta.$$ *(2)*

(c) Using the iterative formula $\theta_{n+1} = 1.5 \sin\theta_n$ and $\theta_1 = 1.4$, calculate the value of θ for which the area of the triangle OAB is twice the area of the segment. Give your answer correct to 3 decimal places. *(3)*

(7)

6 In a crystal structure one of the planes of the crystal has equation

$$2x - 3y + z = 5.$$

An X-ray beam passes through the crystal along the line having equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Find the angle between the X-ray beam and the given plane. *(8)*

(8)

7 (a) The rate of change of the radius (r cm) of a spherical pebble with time (t years), is modelled by the differential equation

$$\frac{dr}{dt} = k, \text{ where } k \text{ is a constant.}$$

If initially the radius is 5 cm and after 20 years it is 4 cm, solve the differential equation and hence find an expression for r in terms of t. *(4)*

(b) Assuming the model $\frac{dr}{dt} = k$, use the chain rule to show that the rate of loss of the volume of the pebble is proportional to its surface area at time t.

(Surface area of a sphere $= 4\pi r^2$, Volume $= \frac{4}{3}\pi r^3$) *(4)*

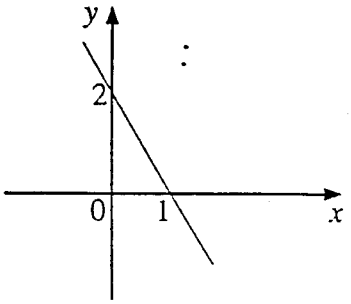
(8)

Appendix 3.1 continued

Section B Answer all questions

8 (a) Describe fully the sequence of transformations that maps the graph of $y = \sin x°$ to that of $y = 5\sin(x + 40)°$. *(3)*

(b) A sketch graph for the function $g(x)$ is shown.



Draw carefully, indicating the intercepts with the axes in each case, separate sketch graphs for the functions

(i) $g(2x)$,

(ii) $g(-x)$,

(iii) $g^{-1}(x)$. *(6)*

(9)

9 The function $f(x) = 3x^3 - 11x^2 - 95x + 175$ has 3 linear factors.

(a) Find $f(5)$ and use your result to explain why $(x - 5)$ is **not** a factor of $f(x)$. *(2)*

(b) The function $f(x)$ may be written in the form

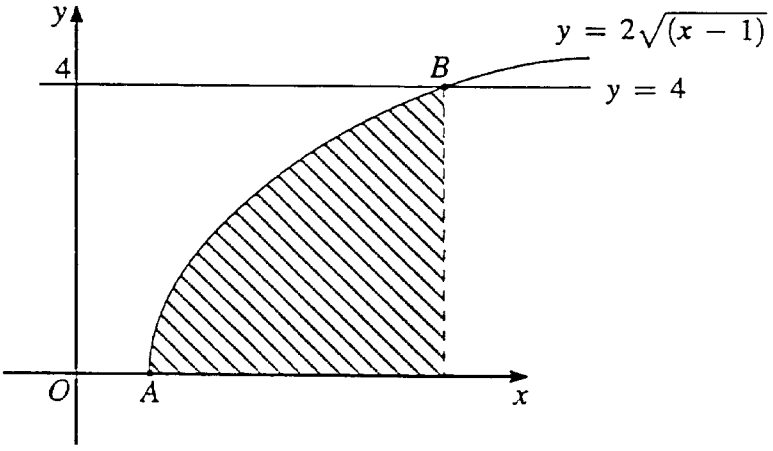
$$f(x) = (x + 5)(ax^2 + bx + c).$$

Find the values of a, b and c and hence write $f(x)$ as the product of its three linear factors. *(6)*

(c) Find the values of x for which $f(x) \geqslant 0$. *(4)*

(12)

10 The graph of $y = 2\sqrt{(x - 1)}$ is shown in the sketch, together with the line $y = 4$.



(a) Find the coordinates of the points A and B. *(2)*

(b) Find the **exact** value of the shaded area shown. *(6)*

(c) Calculate the **exact** volume of the solid formed by rotating the shaded area through 360° about the x axis, leaving your answer in terms of π. *(6)*

(14)

Appendix 3.1 continued

11 (a) Express the function $2\sin x° + \cos x°$ in the form $R\sin(x + \alpha)°$, stating the values of R and α. Using these values, write down the coordinates of the maximum turning point on the graph of

$$2\sin x° + \cos x° \text{ for } 0 \leqslant x \leqslant 90.$$ *(5)*

(b) Express $3\cos 2x + \sin x$ in terms of $\sin x$. Hence calculate all of the values of x between 0 and 360 which satisfy the equation

$$3\cos 2x° + \sin x° = 1.$$ *(8)*

(13)

12 Obtain the first and second derivatives of the function $e^{2x} - \cos x$. Hence, using the Maclaurin Series, show that $2x + \frac{5}{2}x^2$ is a quadratic approximation for $e^{2x} - \cos x$ around the origin.

A quadratic approximation for $\cos 2x$ is $1 - 2x^2$. Use these two approximations to find the approximate solution to $e^{2x} - \cos x = \cos 2x$ in the interval 0 to 1. *(10)*

(10)

13 (a) A curve is defined by the parametric equations $x = 5\cos t$, $y = 2\sin t + 1$.

Find $\frac{dy}{dx}$ at the point on the curve where $t = \frac{\pi}{3}$ *(4)*

(b) Use an **exact** method to evaluate $\int_0^{\frac{\pi}{2}} x\cos x\,dx$, leaving your answer in terms of π. *(5)*

(c) If $f(x) = \frac{3}{x(x+1)}$, write $f(x)$ in the form $f(x) = \frac{A}{x} + \frac{B}{x+1}$ where A and B are constants to be found.

Hence find $\int f(x)\,dx$. *(5)*

(14)

END OF QUESTIONS

Appendix 3.2 Changes to the A-level syllabus

We present a summary of the extent to which new A-level syllabuses for 1996 cover those topics in the previous A-level core which are not in the new A-level core. The analysis is presented in the form of a table. This may conceal some detailed distinctions, but is done for ease of reference which a more discursive treatment would not provide. Each topic is indicated by a simple Y or a dash, showing whether or not the topic is part of the compulsory element of the board's syllabus, taken by all the candidates irrespective of what award the are entered for, or which combination of modules they take. In some cases a topic may be an optional part of the syllabus, but it cannot be assumed to have been covered by all candidates. This appendix is a slightly revised version of the one to be found in the recent LMS/IMA/RSS report.

Topic	A	B	C	D	E	F	G	H	I	J	K
Rational functions	Y	-	Y	-	-	Y	-	Y	-	Y	-
Partial fractions	Y	Y	Y	-	-	Y	Y	Y	-	Y	Y
Binomial for $\|x\| < 1$	-	Y	-	-	-	-	Y	Y	Y	-	-
Six trigonometric functions	Y	Y	Y	-	-	Y	Y	Y	Y	Y	Y
Sine and cosine rules	Y	Y	Y	Y	-	Y	Y	Y	Y	Y	Y
$a \cos q + b \sin q = r \cos(q+a)$	-	Y	-	-	-	Y	Y	Y	Y	-	-
General soln of trig equation	-	-	-	-	Y	Y	-	-	-	-	-
Small trig approximations	-	Y	-	-	-	Y	-	-	-	-	-
Inverse trig functions	-	Y	Y	-	Y	Y	Y	Y	Y	-	Y
Implicit differentiation	-	-	Y	Y	Y	Y	Y	Y	-	Y	Y
Parametric differentiation	Y	-	Y	Y	Y	Y	Y	Y	Y	Y	Y
Normals	Y	Y	Y	-	Y	Y	Y	Y	Y	Y	-
Small increments	-	-	-	-	-	-	-	-	-	-	-
$\int 1/(1+x^2), \int 1/\sqrt{(1-x^2)}$	-	-	Y	-	-	-	-	-	-	-	Y
Volumes of revolution	-	-	-	Y	-	Y	Y	Y	Y	Y	-
Vectors	-	Y	-	Y	Y	Y	Y	Y	-	-	-
Scalar product	-	Y	-	Y	Y	Y	Y	Y	-	-	-
Vector equation of a line	-	Y	-	Y	Y	Y	Y	-	-	-	-

A: AEB
B: MEI (O & C)
C: NEAB
D: NUFFIELD (OXF)
E: OXFORD
F: O & C
G: SMP 16-19 (NEAB)
H: UCLES (Linear)
I: UCLES (Modular)
J: ULEAC
K: WJEC

This table was taken from Howson, G (1996) A-Level - some considerations in Gresham Special Lecture (1996)

UNIVERSITY OF LONDON
SCHOOL EXAMINATIONS BOARD

General Certificate of Education Examination

JUNE 1986 ADVANCED LEVEL.

Subject Title and Subject Code No.	**Mathematics Syllabus B 371**
	Pure Mathematics 405
	Pure Mathematics with Statistics 420
Paper No./Title	**Paper 2**

Two and a half hours

Full marks may be obtained for answers to ALL questions.
Mathematical formulae and tables are provided.
In calculations you are advised to show all the steps in your working, giving your answer at each stage.

Answer as many questions as you can.

1. Find the range (image set) of the function f, where
$$f: x \longmapsto 5\cos x + 12\sin x, \quad x \in \mathbb{R}.$$
(3 marks)

2. Show that $\sum_{r=1}^{2n} (2r-1)^2 = \frac{1}{3}n(16n^2-1)$.
(4 marks)

3. Evaluate $\int_0^{\pi} x \sin 2x \, dx$. **(4 marks)**

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4. Taking $\pi/2$ as a first approximation to a root of the equation $3\sin x - 2x = 0$, apply the Newton–Raphson procedure to find two further approximations, giving your answers to 3 decimal places. **(5 marks)**

5. A function f is defined by
$$f: x \longmapsto 1 - \frac{1}{x}, \quad x \in \mathbb{R}, \quad x \neq 0, \quad x \neq 1.$$
Find (*a*) $ff(x)$,
(*b*) $fff(x)$,
(*c*) $f^{-1}(x)$. **(5 marks)**

6. Find, in radians, the general solution of the equation
$$\cos 3\theta = \cos 2\theta.$$
(5 marks)

7. The four distinct points B_1, B_2, B_3 and B_4 lie on a straight line AB and the three distinct points C_1, C_2, C_3 lie on a second straight line AC. Given that none of the other seven points coincides with the point A, determine the number of different triangles that can be formed with vertices selected from
(*a*) the points B_1, B_2, B_3, B_4, C_1, C_2, C_3,
(*b*) the points B_1, B_2, B_3, B_4, C_1, C_2, C_3 and A.
(5 marks)

8. Find the value of the constant k so that the polynomial $P(x)$, where
$$P(x) \equiv x^2 + kx + 11,$$
has a remainder 3 when it is divided by $(x-2)$.
Show that, with this value of k, $P(x)$ is positive for all real x, and sketch the curve $y = 1/[P(x)]$.
(6 marks)

9. Find the modulus and argument of each of the complex numbers z_1 and z_2, where

$$z_1 = 1 + i, \quad z_2 = \sqrt{3} - i.$$

Hence, or otherwise, show that

$$\arg(z_1/z_2) = 5\pi/12.$$

(6 marks)

---

10. An arithmetic series is such that the sum of its first ten terms is 20, and the sum of its first twenty terms is 10. Find the sum of its first forty terms. **(6 marks)**

---

11. Referred to the origin O the points A, B and C have position vectors $(12a\mathbf{i} - 7a\mathbf{j} + 5a\mathbf{k})$, $(-8a\mathbf{i} - 2a\mathbf{j} + 10a\mathbf{k})$ and $(4a\mathbf{i} + 13a\mathbf{j} - 11a\mathbf{k})$ respectively, where a is constant. Show that the vector $(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ is perpendicular to the plane ABC.

Hence, or otherwise, find a cartesian equation of the plane ABC. **(7 marks)**

---

12. Estimate the value of the integral

$$\int_{0{\cdot}01}^{0{\cdot}49} \frac{1}{1 + 2\sqrt{x}}\,dx$$

by using the trapezium rule with three ordinates, giving your answer to 2 decimal places.

Using the substitution $u^2 = x$, or otherwise, obtain the exact value of the integral. **(9 marks)**

---

13. A circle, centre O and radius a, has AB as a diameter and C is a point on AB produced such that $BC = a$. Points P and Q lie on the circle and $PC = QC$. Given that angle $POC = \theta$, show that L, the perimeter of the area enclosed by the lines CP, CQ and the arc PAQ, is given by

$$L = 2a\pi - 2a\theta + 2a(5 - 4\cos\theta)^{1/2}.$$

Show that, when a is constant and θ varies, there is a stationary value of L when $\theta = \pi/3$.

(9 marks)

---

14. Express $\dfrac{x}{(x+1)(x+2)}$ in partial fractions.

Solve the differential equation

$$(x + 1)(x + 2)\frac{dy}{dx} = x(y + 1)$$

for $x > -1$, given that $y = \frac{1}{2}$ when $x = 1$. Express your answer in the form $y = f(x)$. **(10 marks)**

---

15. Using the same axes, sketch the curves

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{x}{x + 2}$$

State the equations of any asymptotes, the coordinates of any points of intersection with the axes and the coordinates of any points of intersection of the two curves.

Hence, or otherwise, find the set of values of x for which

$$\frac{1}{x} > \frac{x}{x + 2}$$

(13 marks)

---

16. Use the binomial expansion to express $x^4(1 - x)^4$ as a polynomial in x.

Hence, or otherwise, verify that

$$x^4(1 - x)^4 \equiv (1 + x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4.$$

Use this result to evaluate

$$\int_0^1 \frac{x^4(1 - x)^4}{1 + x^2}\,dx$$

and deduce that $\pi < 22/7$. **(13 marks)**

Appendix 3.4 London Examinations Pure Mathematics—1986

London EXAMINATIONS
UNIVERSITY OF LONDON EXAMINATIONS & ASSESSMENT COUNCIL

GCE

Friday 26 May 1995 – 1st Afternoon Session

Advanced Level/Advanced Supplementary

Pure Mathematics P1 (New Syllabus)

Time: 1 hour 30 minutes

Instructions to Candidates

Full marks may be obtained for answers to ALL questions.

In the boxes on the Answer Book, write your centre number, candidate number, the syllabus title (Pure Mathematics), syllabus number (6405), the paper number (P1), your surname and initials, signature and date.

Information for Candidates

A booklet 'Mathematical Formulae including Statistical Formulae and Tables' is provided.

In calculations you are advised to show all the steps in your working, giving your answer at each stage.

This paper has 9 questions.

Appendix 3.4 continued

1. Find, in degrees to 1 decimal place, the values of x which lie in the interval $-180° \leqslant x \leqslant 180°$ and satisfy the equation

$$\sin 2x = -0.57.$$

(6 marks)

---

2. The straight line passing through the point $P(2, 1)$ and the point $Q(k, 11)$ has gradient $-\frac{5}{12}$.

(a) Find an equation of the line in terms of x and y only.

(b) Determine the value of k.

(c) Calculate the length of the line segment PQ.

(8 marks)

---

3. Show that the elimination of x from the simultaneous equations

$$x - 2y = 1;$$
$$3xy - y^2 = 8,$$

produces the equation

$$5y^2 + 3y - 8 = 0.$$

Solve this quadratic equation and hence find the pairs (x, y) for which the simultaneous equations are satisfied.

(10 marks)

4.

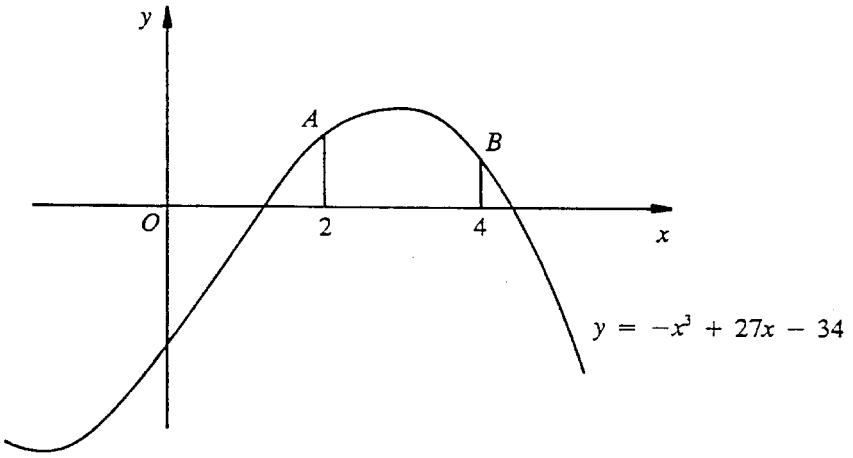


Fig. 1

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = -x^3 + 27x - 34.$$

(a) Find $\int f(x)\, dx$.

The lines $x = 2$ and $x = 4$ meet the curve at points A and B as shown.

(b) Find the area of the finite region bounded by the curve and the lines $x = 2$, $x = 4$ and $y = 0$.

(c) Find the area of the finite region bounded by the curve and the straight line AB.

(11 marks)

Appendix 3.4 continued

5.

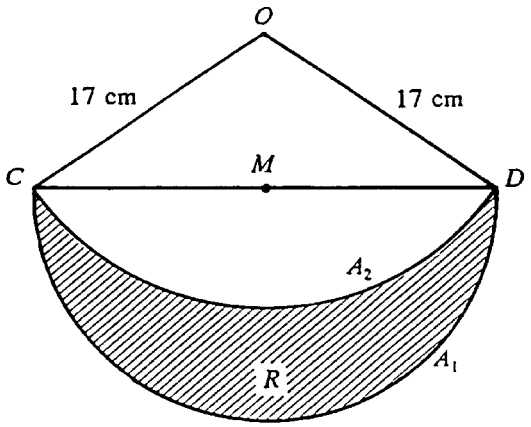


Fig. 2

Figure 2 shows the triangle OCD with $OC = OD = 17$ cm and $CD = 30$ cm. The mid-point of CD is M. With centre M, a semicircular arc A_1 is drawn on CD as diameter. With centre O and radius 17 cm, a circular arc A_2 is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places,

(*a*) the area of the triangle OCD,

(*b*) the angle COD in radians,

(*c*) the area of the shaded region R.

(12 marks)

---

6. The nth term of a sequence is u_n, where $u_n = 95\left(\frac{4}{5}\right)^n$, $n = 1, 2, 3, \ldots$.

(*a*) Find the values of u_1 and u_2.

Giving your answers to 3 significant figures, calculate

(*b*) the value of u_{21},

(*c*) $\sum_{n=1}^{15} u_n$.

(*d*) Find the sum to infinity of the series whose first term is u_1 and whose nth term is u_n.

(12 marks)

---

7. A large tank in the shape of a cuboid is to be made from 54 m^2 of sheet metal. The tank has a horizontal rectangular base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

(*a*) Show that the volume, V m^3, of the tank is given by

$$V = 18x - \tfrac{2}{3}x^3.$$

(*b*) Given that x can vary, use differentiation to find the maximum value of V.

(*c*) Justify that the value of V you have found is a maximum.

(12 marks)

Appendix 3.4 continued

8.

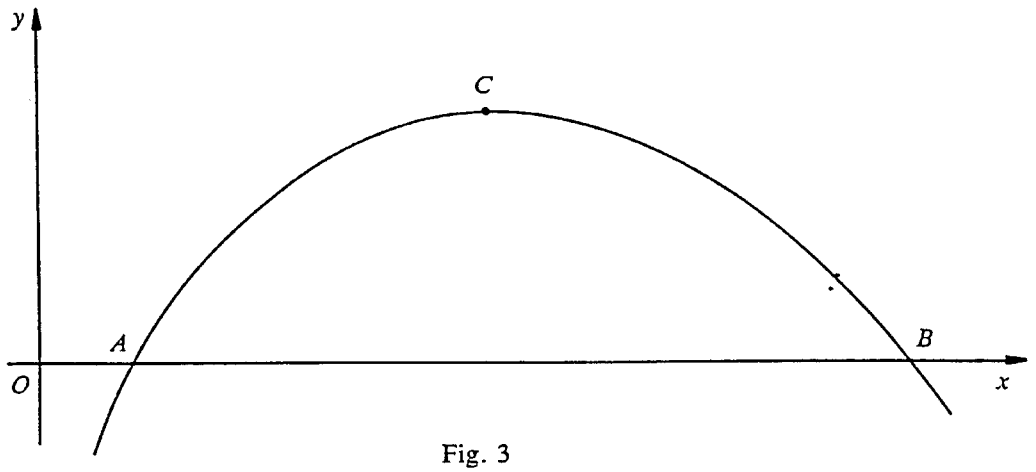


Fig. 3

The function f is defined for positive real values of x by

$$f(x) = 12 \ln x - x^{\frac{3}{2}}.$$

Figure 3 shows a sketch of the curve with equation $y = f(x)$. The curve crosses the x-axis at the points A and B. The gradient of the curve is zero at the point C.

(a) By calculation, show that the value of x at the point A lies between 1.1 and 1.2.

The value of x at the point B lies in the interval $(n, n + 1)$, where n is an integer.

(b) Determine the value of n.

(c) Show that $x = 4$ at the point C and hence find the greatest positive value of $f(x)$, giving your answer to 2 decimal places.

(d) Write down the set of values of x for which $f(x)$ is an increasing function of x.

(14 marks)

---

9. The functions f and g are given by

$$f : x \longmapsto 3x - 1, \quad x \in \mathbb{R},$$

$$g : x \longmapsto e^{\frac{x}{2}}, \quad x \in \mathbb{R}.$$

(a) Find the value of fg(4), giving your answer to 2 decimal places.

(b) Express the inverse function f^{-1} in the form $f^{-1} : x \longmapsto \ldots$.

(c) Using the same axes, sketch the graphs of the functions f and gf. Write on your sketch the value of each function at $x = 0$.

(d) Find the values of x for which $f^{-1}(x) = \dfrac{5}{f(x)}$.

(15 marks)

12. The sets P and Q are defined as follows:

$P = \{x : (10 - 2x)^3 = 2x^3\}$,

$Q = \{x : x(2 + \sqrt{2}) = 10\}$.

Show that $Q \subset P$, and explain why $Q \neq P$.

Hence, or otherwise, solve the equation

$(10 - 2x)^3 = 2x^3$ ..... (1)

giving the roots correct to two places of decimals.

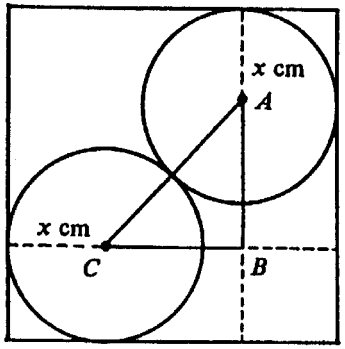


Fig. 1

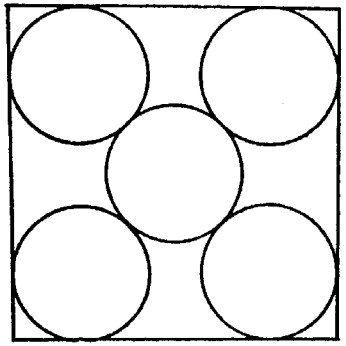


Fig. 2

Fig. 1 represents a plan view of two cylindrical cans of radius x cm, which fit tightly into a square box of side 10 cm. By considering the triangle ABC, or otherwise, show that x is a member of set P, above, and write down the value of x.

Fig. 2 shows five smaller cans fitting tightly into the same box. Form, but do not solve, an equation of the same type as (1), above, whose solution would give the radius of one of these smaller cans.

10. (i) The first term of an arithmetic progression is $(x^2 + 1)$ and the second term is $(x + 1)^2$. If the sum of the first twelve terms is 516, and x is negative, find the value of x.

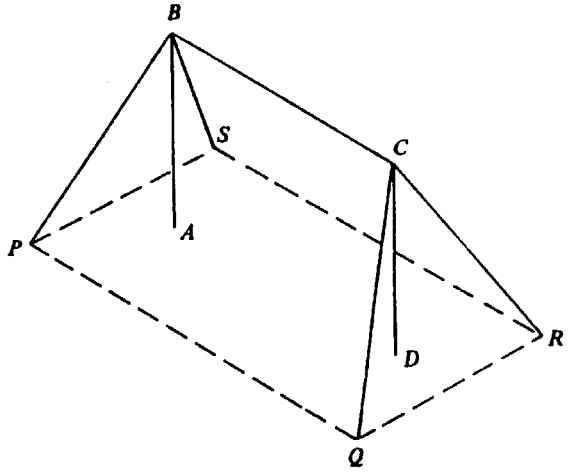
(ii) The second term of a geometric progression is 2 and the fifth term is $6\frac{3}{4}$. Calculate

(*a*) the common ratio,

(*b*) the first term,

(*c*) the sixth term.

13.



Part of the framework of a tent consists of two vertical poles AB, CD each of length 1·80 m, connected by a horizontal pole BC of length 2·20 m. The framework is held in position by four equal ropes BP, BS, CQ, CR with their lower ends forming a rectangle $PQRS$ on the same horizontal plane as A and D.

Given that $PQ = 3{\cdot}00$ m and $QR = 1{\cdot}60$ m, calculate

(*a*) the length of one of the equal distances AP, AS, DQ, DR,

(*b*) the inclination of each of the ropes to the horizontal,

(*c*) the length of each rope,

(*d*) the angle between the plane $PBCQ$ and the horizontal.

8. Express as a single fraction in its simplest form

$$\frac{1}{xy - x^2} - \frac{1}{y^2 - xy}$$

10. When a certain typist was under training, it was observed that the average number of mistakes she made per hundred words typed varied with the number of words she typed per minute, as shown in the following table:

Words per minute (x)	20	30	40	50	60	70
Mistakes per 100 words (y)	5	7·5	10	12·5	15	17·5

Show that all these results are consistent with a law of the form $y = kx$, and find k.

If the time taken to correct each mistake is 10 seconds, show that the time, T seconds, taken to type each 100 words, including correcting mistakes, is given by

$$T = \frac{6000}{x} + \frac{5x}{2}.$$

Either (*a*) find $\frac{dT}{dx}$ and the value of x for which $\frac{dT}{dx} = 0$, and indicate on a sketch-graph of $x \to T$ the point on the graph at which x has this value. Explain the significance of this value of x.

Or (*b*) draw an accurate graph of $x \to T$ for values of x from 20 to 70, and use the graph to estimate the value of x for which the gradient of the graph is zero. Explain the significance of this value of x.

31. A solid is formed by rotating completely about the x-axis the curve $y = x^2 + x - 6$ between the points where the curve cuts the x-axis. The volume of the solid is

A $\pi \int_{-2}^{3} y \, dx$

B $\pi \int_{-3}^{2} y \, dx$

C $\pi \int_{2}^{3} y^2 \, dx$

D $\pi \int_{-3}^{2} y^2 \, dx$

E $\pi \int_{-3}^{-2} y^2 \, dx$

1. (i) In a sale the price of an article is reduced from £18·75 to £15. Calculate the reduction as a percentage of the original selling price.

(ii) Find, without using tables, the value of $9^{\frac{1}{2}} \times 27^{-\frac{2}{3}}$.

(iii) Calculate the value of $\int_1^2 (3x^2 + 2x) \, dx$.

(iv) A variable chord of length 6 cm is drawn in a fixed circle with centre O and radius 5 cm. Show that the locus of the mid-point of the chord is a circle and state its centre and radius.

2. Show that the pair of inequalities

$$2 \leqslant 2x + 1 \leqslant 6 - 3x$$

is satisfied by the same set of values of x as is the inequality

$$2x^2 - 3x + 1 \leqslant 0.$$

If x is a member of this set, and $x = \sin \theta$, find the set of possible values of θ, given that $0° < \theta < 180°$.

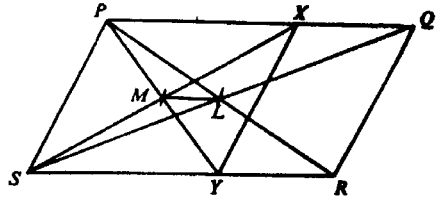
4. Solve the simultaneous equations $x^2 - y^2 = 22$, $x - y = 11$.

10. (i) Given that $x - 3$ is a factor of the expression $x^3 + kx^2 - 5x + 6$, calculate the value of k. Hence determine the other factors of the expression.

(ii) Solve for x and y the equations

$$x - y = 4,$$
$$x^2 - 4x + y^2 - 2y = 0.$$

39.



PQRS is a parallelogram with $PS = 5$ cm, $SR = 9$ cm and the diagonals intersecting at L. *PXYS* is a rhombus with its diagonals intersecting at M. The length of ML, in cm, is

A 2

B $2\frac{1}{2}$

C $3\frac{1}{2}$

D 4

E not determined from the information given

Appendix 3.6 International Baccalaureat/A-level equivalence

It has been agreed by HMC that Sevenoaks School may convert International Baccalaureate results into A level terms by means of the conversion formula printed below. Although this table is not yet recognized by the DFE we believe it to be fair and as accurate as it can be bearing in mind the different nature of the two examinations. This formula has been used in August for the last three years with HMC approval when we have submitted results to the national press for their league tables.

HIGHER LEVEL IB GRADE	EQUIVALENT A LEVEL GRADE	POINTS VALUE	SUBSID LEVEL IB GRADE	EQUIVALENT A/S LEVEL GRADE	POINTS VALUE
7	A	10	7	A	5
6	A/B	9	6	A/B	4.5
	B	8		B	4
5	B/C	7	5	B/C	3.5
	C	6		C	3
	D	4		D	2
4	D/E	3	4	D/E	1.5
	E	2		E	1
3	(E)/N	0.5	3	(E)/N	0.25
2	N	0	2	N	0

Notes.

1. Each IB Diploma candidate gains points for his/her three Higher level subjects and the best one of his/her Subsidiary subjects. Thus the Diploma package is regarded as the equivalent of three A levels and one AS level in terms of workload.

2. It is in our experience more difficult to achieve a top grade in certain IB subjects than in others. A significant number of our candidates who would have gained an A grade in A level **Maths or Science** manage only a 5 in Higher level IB. The same is true for *bona fide* students of **Modern Languages** where the IB standard is kept artificially high by the large numbers of bilingual candidates who enter for the same examination. We would be grateful if Universities could bear these differences in mind when making conditional offers to IB candidates.

3. It should be noted that other IB schools in the UK regard this conversion table as ungenerous to the candidate.

Richard Russell
Director of Studies 24/1/96

Appendix 3.7 Example of A-level investigation

The triangular numbers are 1, 3, 6, 10 etc. i.e. ½n (n+1)
The square numbers are 1,4,9,16 etc. i.e. m^2

Which numbers are both triangular and square? The first two are 1 and 36.

The structure underlying the problem is revealed by showing that it boils down to finding the integer solutions to the equations $x^2 - 2y^2 = \pm 1$. The *link* with this is easy enough, but actually showing that the complete set of solutions is given by

$$(x, y) = \{(a_n, b_n) \mid n = 1,2,\cdots\}, \text{ where } a_n + b_n\sqrt{2} = \left(1 + \sqrt{2}\right)^n,$$

is rather more difficult. A relatively elementary treatment can be given at undergraduate level, but even this is more 'naturally' understood in the context of the group-theoretic structure of the group of units in the ring of integers of a finite algebraic number field. The set of solutions to $x^2 - my^2 = 1$ (where m is a positive integer which is not a square) corresponds to a special case where the group is cyclic.

From the nature of the link, it follows that the nth square triagular number is $(a_n b_n)^2$, where $a_n + b_n\sqrt{2} = \left(1 + \sqrt{2}\right)^n$.

Using the binomial expansion

$$\left(1 + \sqrt{2}\right)^n = 1 + n\sqrt{2} + \binom{n}{2}\left(\sqrt{2}\right)^2 + \binom{n}{3}\left(\sqrt{2}\right)^3 + \cdots + \left(\sqrt{2}\right)^n$$

then gives that the nth square triangular number is

$$\left[\sum_{r=0}^{\frac{n}{2}} 2^r \binom{n}{2r}\right]^2 \left[\sum_{r=0}^{\frac{n}{2}-1} 2^r \binom{n}{2r+1}\right]^2, \text{ if } n \text{ is even;}$$

and

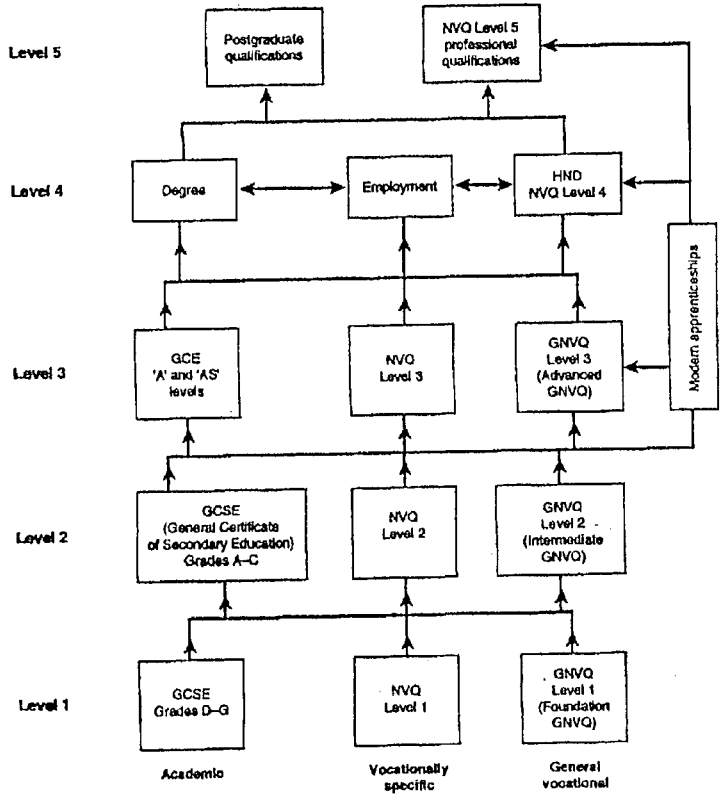
$$\left[\sum_{r=0}^{\frac{n-1}{2}} 2^r \binom{n}{2r}\right]^2 \left[\sum_{r=0}^{\frac{n-1}{2}} 2^r \binom{n}{2r+1}\right]^2, \text{ if } n \text{ is odd.}$$

B THE CURRENT SYSTEM

Courses and qualifications

17. Figure 1 shows the major categories of awards from 14 upwards in England. It shows clearly the "three pathways" within which provision is currently channelled, and how the five levels of the NVQ framework can be "read across" to other qualifications — for instance, NVQ Level 3 is regarded as equivalent to 2 A-levels.

Figure 1: The major categories of awards from 14 upwards

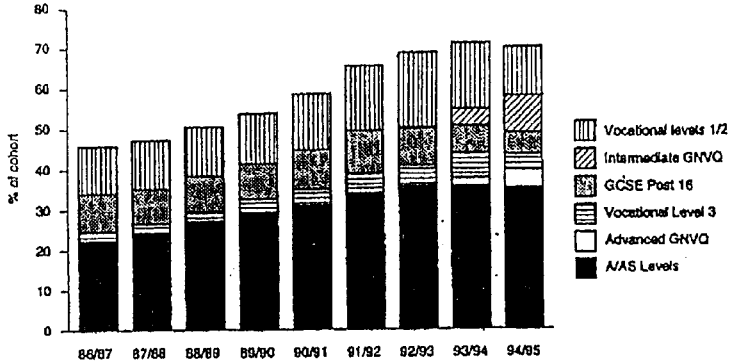


(Source: A guide to Further Education in England and Wales, Leonard Cantor, IoloRoberts and Beryl Pratley, 1995)

From House of Commons Enquiry
14–19 Ed. Published End Feb 96

18. Figure 2 shows participation by 16 year olds in each of the different routes. Information on participation levels at 16, 17 and 18 can be found in Annex 2.

Figure 2: Full-time participation by course at 16



% of cohort

	86/87	87/88	88/89	89/90	90/91	91/92	92/93	93/94	94/95
A/AS Levels	21.9	23.7	27	29.4	31.8	34.2	35.5	36.7	36.2
Advanced GNVQ							0.1	1.9	4.6
Vocational Level 3	3.1	3	3	3.7	4.3	5.3	6.1	6.2	4.0
GCSE Post 16	9.7	8.9	8.9	8.9	9.8	11	9.5	6.8	5.0
Intermediate GNVQ							0.1	4.3	9.7
Vocational Level 2	11.5	12.2	12.1	12.6	12.9	15.6	17.9	15.9	11.5

(Source: Learning for the Future, Working Paper 7, Post-compulsory education and training: Statistical trends, Ken Spours, November 1995)

GCSEs

19. The General Certificate of Secondary Education was introduced in 1986, replacing the GCE O-level and the Certificate of Secondary Education (CSE). It is the main certificate for recognising achievement at 15 and 16 by school students, although many also take it after 16 — about one per cent of the 17 year old cohort in 1994-95.[13] Some of the GCSE courses now available have a vocational aspect, such as the GCSE in Business Studies taken by 117,000 students in 1994.[14] A wide range of subjects is available. The levels of award range from A* down to G. In 1994, 43.3 per cent of 16 year old pupils obtained five or more GCSEs at Grade C or above.[15] GCSEs are assessed by a combination of terminal assessment and coursework. GCSE syllabi are designed by the various examining boards; the courses and qualifications have to be validated by the School Curriculum and Assessment Authority (SCAA) (see paragraph 75 below). SCAA also maintains a range of mechanisms for ensuring quality and consistency: for instance, GCSEs in National Curriculum subjects have to reflect statutory programmes of study, while general regulations govern other subjects. The GCSE Code of Practice also helps in this process.[16]

[13] *Learning for the Future*, Working Paper 7, p.16.
[14] HC 649, p.7.
[15] *Ibid.*
[16] Evidence, pp.153, 161-62, 164.

Appendix 4.2 Minimum requirements of HE Courses (Lord, Wake & Williams, 1995)

Conclusions

A In preparation for any degree course, the greater number of mathematics *content groups* met by a student, the greater the number of university courses for which he or she would be appropriately prepared.

Each *content group* is a collection of syllabus items, for example, the *content group Number 2* consists of : (use the modulus sign; use number bases other than base 10; understand rational and irrational numbers; calculate absolute and relative errors; use and calculate numbers expressed in standard index form; use the laws of indices; use the laws of logarithms). (Full descriptions of the *content groups* used are given on the questionnaire in Appendix B.)

Table 7 below shows, for each degree course, the *content groups* ranked in descending order of median rating. The rating indicates the proportion of the content that is a pre-requisite for that course, ranging from 0 (none) to 3 (all). The *content groups* with dark shading had a median rating ≤ 2, suggesting that more than half the universities surveyed required all or most of this content, and those with light shading had a median rating ≥ 1 but less than 2, suggesting than more than half the universities required some of this content. The unshaded *content groups* had a median rating < 1, suggesting that the majority of the universities required none of this content.

Biology	Building & construction	Business & Economics	Chemistry	Civil Engineering	Computing	Electrical Engineering	Manufact-uring	Mechanical Engineering	Physics
Number 1	Number 1	Number 1	Number 1	Number 1	Number 2	Number 1	Number 1	Number 1	Number 1
Number 2	Mensuration	Algebra 1	Algebra 1	Algebra 1	Algebra 1	Algebra 1	Algebra 1	Algebra 1	Algebra 1
Statistics 1	Trig 1	Functions 1	Functions 1	Functions 1	Number 2	Trig 1	Trig 1	Trig 1	Trig 1
Mensuration	Algebra 1	Applying	Calculus 1	Calculus 1	Algebra 2	Trig 2	Trig 2	Trig 2	Trig 2
Fractions 1	Number 2	Number 2	Number 2	Number 2	Statistics 1	Number 2	Mensuration	Mensuration	Number 2
Algebra 1	Fraction 1	Algebra 2	Algebra 2	Trig 1	Trig 1	Functions 1	Number 2	Number 2	Calculus 1
Algebra 2	Algebra 2	Functions 2	Statistics 1	Calculus 3	Trig 2	Calculus 1	Functions 1	Calculus 1	Calculus 3
Statistics 3	Trig 2	Statistics 1	Trig 1	Mensuration	Calculus 1	Calculus 3	Kinematics	Calculus 3	Mensuration
Functions 2	Functions 2	Mensuration	Calculus 3	Trig 2	Calculus 3	Mensuration	Calculus 1	Functions 1	Kinematics
Calculus 1	Calculus 1	Trig 1	Functions 2	Algebra 2	Mensuration	Kinematics	Functions 2	Kinematics	Functions 1
Calculus 3	Calculus 2	Trig 2	Mensuration	Functions 2	Functions 1	Functions 2	Vectors	Functions 2	Functions 2
Applying	Statistics 1	Calculus 1	Applying	Calculus 2	Functions 2	Algebra 2	Calculus 3	Algebra 2	Algebra 2
Prob 1	Statistics 2	Calculus 2	Trig 2	Kinematics	Comp. No.	Calculus 2	Algebra 2	Calculus 2	Calculus 2
Trig 1	Applying	Statistics 2	Calculus 2	Statistics 1	Vectors	Comp. No.	Statistics 1	Calculus 4	Comp. No.
Trig 2	Prob 1	Prob 1	Prob 1	Applying	Applying	Calculus 4	Calculus 2	Vectors	Calculus 4
Calculus 2	Kinematics	Kinematics	Kinematics	Calculus 4	Matrices	Vectors	Calculus 4	Statistics 1	Vectors
Calculus 4	Prob 2	Prob 2	Calculus 4	Comp. No.	Prob 1	Statistics 1	Comp. No.	Comp. No.	Statistics 1
Comp. No.	Calculus 3	Calculus 3	Statistics 2	Matrices	Kinematics	Applying	Applying	Applying	Applying
Matrices	Calculus 4	Calculus 4	Prob 2	Vectors	Calculus 2	Matrices	Prob 1	Prob 1	Matrices
Vectors	Comp. No.	Comp. No.	Comp. No.	Probability 1	Calculus 4	Prob 1	Matrices	Matrices	Prob 1
Xinematics	Matrices	Matrices	Matrices	Statistics 2	Number 1	Prob 2	Prob 2	Prob 2	Statistics 2
Prob 2	Vectors	Vectors	Vectors	Prob 2	Number 1	Statistics 2	Statistics	Statistics 2	Prob 2

Table 7. The ranked content group requirements for each degree course, in descending median order.

reinstated within school mathematics.)

In GNVQ courses mathematics is expected to be taught within the context of realistic situations related to vocational employment. **Consideration should be given to the extent to which it is feasible for algebraic work to be learned within the context of vocational employment.**

5.7 Implications for assessment

Current assessment practices in mathematics tend to place more emphasis on correct answers than on the process of solution and it is the latter which is crucial to algebra. The dominant view seems to be that current assessment practices in England and Wales support pupil learning but, in the case of algebra, the nature of this learning is not clear.

The effects of form of assessment on learning in mathematics needs to be investigated. More attention should be given to syllabus design and assessment in order to promote algebraic activity.

Key Stage tests are driven by the need to evaluate questions in the field and the need to remove aspects of a question which are found to be too difficult. This often results in the original algebraic purpose of a question being removed from the question. **Key Stage tests and GCSE examination questions in England and Wales are often dressed-up within spurious contexts. The aims and effects of these practices need to be critically examined.**

5.8 Implications for the development of curriculum materials

What pupils learn is inextricably linked to what they engage with. When they use textbooks, the presentation of these texts will influence the mathematics learned. This is also the case for computer-based presentation of materials. **Those involved in the presentation of mathematics to pupils, which includes teachers who prepare materials for the classroom either on the board or on paper, need to reflect carefully on the likely learning effects of the presentation they choose.**

The fact that a textbook or CD-Rom sells well and is popular with teachers and pupils does not imply that pupils are learning appropriate mathematics, or that the text is mathematically accurate or internally consistent. The current situation is that how materials are presented is often influenced by factors such as publisher's interests and the market. A mechanism has to be found which enables feedback on what pupils learn from these materials to be taken into account. *This suggests that curriculum materials should be subject to more critical evaluation.*

5.9 Implications regarding new technologies

This was the most difficult and controversial issue for the working group to report on and relates to different views about what mathematical knowledge is, which cannot be the focus of this report.

In England and Wales many changes to the mathematics curriculum have centred around new technologies. This is particularly the case with calculators and graphics calculators. This has had complex and unpredicted effects, such as when primary pupils use 'trial and improvement' with a calculator to solve problems which were intended to teach the idea of inverting arithmetical operations. Banning the calculator from a Key Stage test will not result in pupils changing their well-established method within the test, although it might change a teacher's emphasis in training pupils. Banning pupils from ever using calculators in school is not sensible or practical as they are widely used outside school.

In France, for example, pupils are allowed to use graphics calculators in the Baccalaureat and will use graphical methods to solve a range of problems, where previously they may have only used algebraic methods. However, the whole mathematics curriculum has not changed in order to embrace graphics calculators and other computational tools as appears to be the case in England and Wales.

Work with certain types of symbolic computer environments can support pupils to learn crucial algebraic ideas, for example using symbols to represent quantitative relationships between variables. Work with computers has also shown that, when engaged in solving appropriate problems, pupils can become confident and competent in using symbols to communicate mathematical ideas. *We should try to capitalize on this possibility.*

Computer algebra systems threaten the very nature of school mathematics because they can perform most of the routine algebraic computations which most pupils have always found very difficult and alienating. As their price decreases the vast majority of pupils will have access to such systems.

We recommend the need for more research on what pupils can learn through using algebraic calculators.

Mathematics educators need to become more aware of the complex relationship between learning, the problem being solved and the tools which are available. They need to be more aware of what it is they want pupils to learn in order to decide on when to tell pupils to use a computational tool, when to tell them to use paper and pencil and when to tell them to carry out a process mentally. They need to communicate why they are doing this to the pupils themselves.

Paper technology did not preclude teachers from asking pupils to work mentally. Computer technology should not preclude teachers from asking pupils to work with paper. In the past, before computers were available, novice teachers probably learned a range of practices from expert teachers. However,

technology is changing so rapidly that very few teachers are expert in the area of using new technologies. This is why teacher education is crucial. *In England and Wales the current financial constraints often inhibit even the most motivated teachers from attending courses on the use of computers for teaching and learning mathematics (or, indeed, on mathematics itself).*

We recommend that resources are made available to educate teachers to use new technologies to promote learning of algebra and mathematics.

5.10 Implications for teacher education

Teachers need support and guidance in order to recognize the essential nature of algebraic activity. Work needs to be done to develop materials and courses to achieve this.
We recommend that funding is made available for in-service teacher education. This should include courses for teachers in primary schools, secondary schools and FE colleges. Those involved in both pre-service and in-service training need to engage with the issues raised in this report.

5.11 Implications for decision making

The universities have drawn attention to a problem which needs solving. Solving the problem does not imply returning to traditional forms of teaching but it does imply change. It also implies critical examination of dogmatic views about education which can emanate from both school and university mathematics educators. Already the current debate has provoked productive discussion and collaboration between mathematicians, teachers, teacher trainers and those involved in mathematics education research. It has also provoked a less insular perspective, examining education systems in other parts of the world.

We urge and recommend that more reflection and analysis is built into the system. This requires time. It also implies the need for some body with an overall coordinating responsibility for mathematics education from 5 to 19, including the National Curriculum, assessment, teacher supply and training. We should not 'experiment on the job' with our future populations.

5.12 Tailpiece

Finally the following quote re-emphasizes the role and power of mathematics.

> *To criticize mathematics for its abstraction is to miss the point entirely. Abstraction is what makes mathematics work. If you concentrate too closely on too limited an application of a mathematical idea, you rob mathematicians of their most important tools: analogy, generality and simplicity. Mathematics is the ultimate in technology transfer.*
>
> (Stewart, 1989, p. 291)