Forum 3/ Presentation: Students encountering obstacles using CAS *A developmental-research pilot study*

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Abstract

The paper describes the role of the theoretical framework of Realistic Mathematics Education and Developmental Research in developing and performing an educational experiment. In the experiment a symbolic calculator is introduced in a pre-examination class doing A-level mathematics. The results of the study include the identification of obstacles that students encounter while using computer algebra.

1 Introduction

During the last decade the availability of computer algebra environments has increased dramatically, not in the least because of the development of hand-held symbolic calculators. Many students nowadays have access to powerful computer algebra software, and a further diffusion of Computer Algebra Systems (CAS) is to be expected.

In the mean time, educational researchers and teachers are concerned with the fundamental questions that arise as soon as computer algebra is integrated in the teaching and learning of mathematics. To mention just some:

- How can the use of CAS improve conceptual understanding?
- How can the use of CAS affect the curriculum?
- What is the role of paper-and-pencil skills in a computer algebra environment?
- What prerequisite knowledge and skills are required in order to benefit from the availability of computer algebra?

An overview of relevant questions for educational research can be found in Drijvers (1997).
Many studies have been undertaken to answer (parts of) these questions, sometimes with convincing results. Heid (1988) showed how the development of concepts can precede the learning of techniques. Mayes (1997) gives an overview of the research in this field, whereas Monaghan (1994) discusses the findings of five studies that were carried out in the U.K.

In the Netherlands a study on the integration of graphing calculators in mathematics education was carried out by the Freudenthal Institute (Drijvers & Doorman, 1997). A second study focusing on the role of the symbolic calculator was a natural consequence. The findings of the latter project, that was carried out in 1998, are presented in this paper.

In section 2, I describe the theoretical background of the work. Then, the research questions, the methodology (section 3) and the project settings (section 4) are briefly presented. Section 5 deals with the resequencing of concepts and skills. In section 6 the development of algebraic insight is discussed. Section 7 contains some relevant classroom observations that will lead (section 8) to the identification of obstacles that students encountered while working with the symbolic calculator. A concluding discussion can be found in section 9. Throughout the article I will refer to the theoretical framework to illustrate the interaction between theory and educational experiment.

2 Theoretical framework

2.1 Realistic Mathematics Education

The domain specific theory of Realistic Mathematics Education (RME) forms the theoretical framework for this study. This instruction theory has acquired considerable impact in the Netherlands during recent decades. According to this theory, mathematics is considered a human activity (Freudenthal, 1991). Realistic problem situations should play an important role in the learning process right from the start. Solution procedures are (re)constructed by the students themselves through using problems that have meaning in their reality. Van Reeuwijk (1995) provides the following characteristics of Realistic Mathematics Education, that I will summarize briefly:

- *'real' world*
  Learning of mathematics starts from problem situations that students perceive as real or realistic. These can be real life contexts, but they can also arise from mathematical situations that are meaningful and natural to the students.
- *own productions and constructions*
  Students should have the opportunity to develop their own informal problem solving strategies, that can lead to the construction of solution procedures. This 'bottom-up' reinvention process is guided by the teacher and the instructional materials.
- *mathematization*
  Usually two types of mathematization are distinguished: horizontal mathematization which refers to modelling the problem situation into mathematics, and vertical mathematization, which refers to the process of reaching a higher level of abstraction.
- *interaction*
  Interaction among students and between students and the teacher is important in RME, because discussion and cooperation enhance the reflection that is essential for the reinvention process.
- *integrated learning strands*
  In the philosophy of RME, different mathematical topics should be integrated in one curriculum. The student should develop an integrated view of mathematics, as well as the flexibility to connect the different sub-domains.

An extensive discussion of the theory of realistic mathematics education can be found in Freudenthal (1991), Gravemeijer (1994) and Treffers (1987). Of course, the philosophy and the theory of RME are not rigid structures. They are subject to change and development, that are incited by the findings in educational experiments. This brings us to the second aspect of the framework, the developmental research.

2.2 Developmental Research

The methodology for research on the theory of RME has similar characteristics to the theory itself: In interaction with the 'real-life' classroom situation, the researcher tries to 'reinvent' the theory by means of constructing and developing thought experiments and educational experiments. This so-called 'developmental research' methodology involves a cyclic process of consideration and testing, an alternation of thought experiment and educational experiment. In the thought experiment the designer imagines how the educational process might take place. The thought experiment is then tested in the educational experiment.

Freudenthal put it this way (Freudenthal, 1991, p. 161):

> *Developmental research means: experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.*

Neither the curriculum being tested nor the underlying theory, however, are left unchanged. The experience acquired is immediately used to adjust the instructional materials and the local theory on instruction: development and testing go hand in hand. Gravemeijer (1993) uses the expression ‘feed-forward’ to illustrate the effect that findings in the educational experiment immediately have both on the theory and of the continuation of the experiment.

For a more extensive description and examples of developmental research see Gravemeijer (1994) and De Lange (1987).

3 Research questions and methodology

The research questions for this short study were provided by a committee of the Dutch National Pedagogical Centres, who are in charge of instigating projects that respond to teachers’ concerns.

This paper focuses on the following questions:

1. *Is it possible to ‘resequence’ a course using a CAS, so that concept development precedes the solving techniques and algorithms?*
   Essentially, this is one of the central questions in the well-known study carried out by Heid (1988). Her reports on this point were quite positive.
2. *Can algebraic insight improve because of CAS use?*
   This idea is often defended. But how do we define algebraic insight? How should a CAS be used in order to achieve improvement?
3. *What obstacles do students experience while working with computer algebra?*
   Identifying obstacles might be very useful in determining what prerequisite knowledge and skills are necessary to make meaningful use of computer algebra.

The project consisted of two parts, a descriptive and an experimental part. The descriptive study included a survey of literature and interviews with international experts. The present paper is about the educational experiment that formed the second part of the project. Please do not take the word ‘experiment’ as indicating a well-controlled comparative study; I prefer to call it an explorative pilot study.
Data was gathered by means of participating classroom observations, video-taping and interviews. Complementary to this qualitative data were the results of a pretest, a posttest and a questionnaire.

4 Developing the educational experiment

The educational experiment took place in a pre-examination class doing the equivalent of A-level mathematics. The class consisted of 22 students - 8 female, 14 male - of about 17 years old. As the computer algebra platform, they used the symbolic calculator TI-92. The main reason for that was the practical advantage of not having to go to a computer lab, where PC’s are usually dominating the educational setting. The fact that the students already owned a TI-83 graphing calculator for more than a year made this choice more evident: the similarities between the interfaces of the two machines would facilitate the students’ introduction to the TI-92.
The students received a TI-92 for a four week period. There were four 50-minute mathematics lessons each week. During these lessons, students worked in pairs for a significant part of the time. Bearing in mind the relevance of interaction in the theory of RME, the partners were stimulated to work together and to communicate on what each of them was doing with their ‘personal’ machine. Every lesson, one pair used a TI-92 that was connected to a viewscreen in order to have the

screen video-taped. These pairs were alternated throughout the experiment. A side-effect of this was that the other students could also see what ‘today’s victims’ were doing. During classroom discussions - also very important in the light of interaction and reflection! - the teacher often asked this pair to do the calculations. The teacher himself did not use the machine or the viewscreen during the lessons.

Two instructional units were developed for the experiment: ‘Introduction TI-92’ and ‘Optimization using a symbolic calculator’. The purpose of thr first unit is learning how to perform the most important calculations on the TI-92. In the mean time, some problems focus on specific aspects that one encounters when working with a CAS (but not only then). These aspects are in italics in the ‘flow chart’ of the unit (see figure 1). The investigation task reflects the RME-idea that students need to have room for exploration and for construction in order to build up their own theory. Technology can be helpful there because it frees the student from calculational drudgery. The investigation task resulted in a written report. These reports were presented to the class.

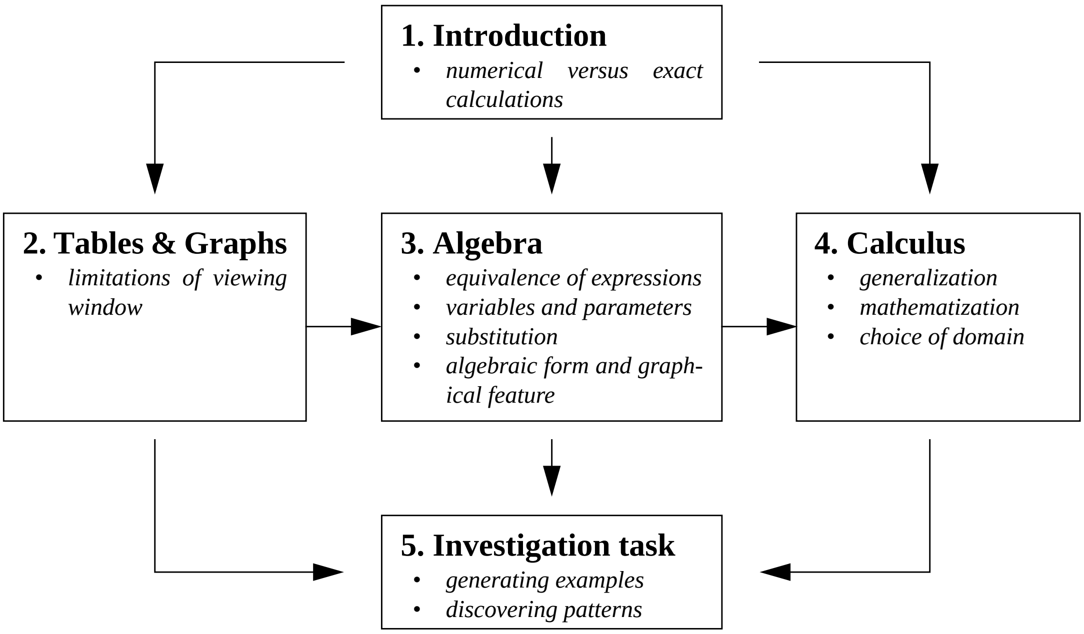


Figure 1: Overview of the unit ‘Introduction TI-92’ with CAS-aspects

Before introducing the second unit, I have to tell you more about the prerequisite knowledge of the students when they entered the experiment. At the start of the school year, the students worked through a unit called ‘Sum and difference, distance and speed’. This unit is about the principles of differentiation and integration, that are developed simultaneously. The concept of the derivative is introduced using the rate of change: the context of speed in a time-distance graph gradually develops into a more generic model for the concept of the derivative. In a similar manner, the integral is introduced as the distance travelled in a time-speed graph. After that, the students learned how the derivative can be used to find extreme values of functions. The only functions they could differentiate manually, however, were power functions. No derivatives of rational or trigonometric functions, nor any rules for differentiation were in the students’ repertoire yet.

Several ideas from the theory of RME guided the development of the unit ‘Optimization using a symbolic calculator’, a revision of an existing unit ‘Optimization using a graphing calculator’:

- A central concept is the modelling of '*Real life*' situations into optimization problems. This involves horizontal *mathematization*.
- A second concept is the relation between the extreme value of a function and the zeros of the derivative. Because the symbolic calculator does the technical part of the work, the student can concentrate on this concept and on the *construction* of a problem solving strategy.
- Optimization problems often can be solved in various ways: numerically, with graphs, with algebra/calculus and with geometry. By means of purposely mixing up all these methods, the unit aims at *integration* of these approaches and increasing flexibility of the student.
- Technology (i.e. hand held computer algebra) can support the *flexibility* in problem solving methods, because it takes over a great part of the manipulative work.

The functions that model the optimization problems cannot be differentiated manually by the students. This can be left to the symbolic calculator, that serves as a 'black box' while doing so.
In figure 2 you see an overview of the unit with brief descriptions of the core of each section in italics.

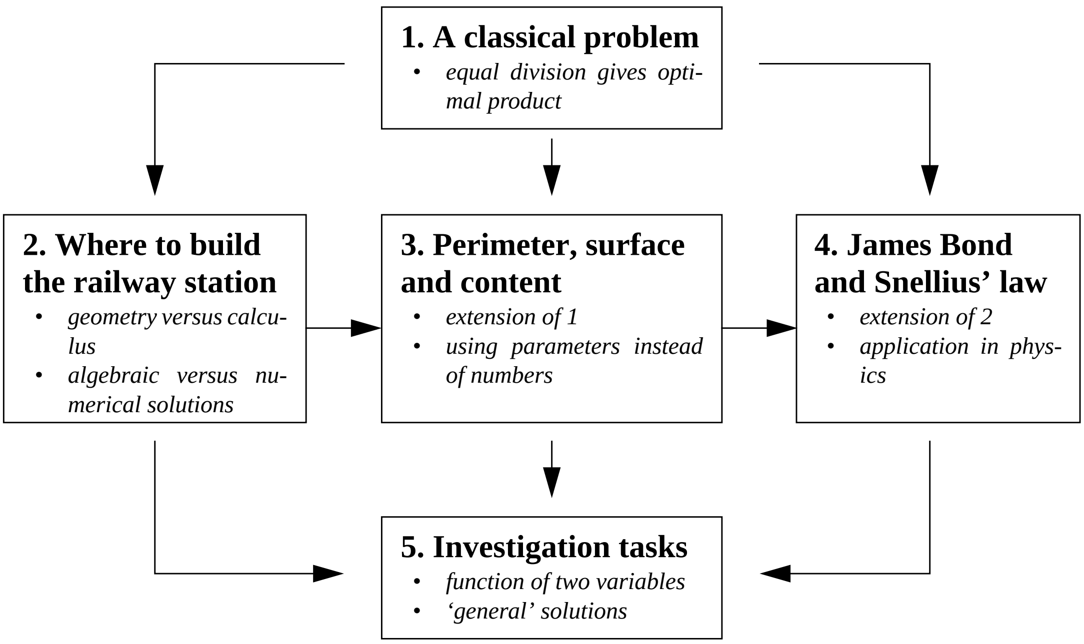


Figure 2: Overview of the unit 'Optimization using a symbolic calculator'

5 Resequencing concepts and skills

The first research question mentioned in section 3 is:

> *Is it possible to 'resequence' a course using a CAS, so that concept development precedes the solving techniques and algorithms?*

As was described above this idea was applied to the modelling of 'real-life' optimization problems and to the development of a problem solving strategy using the zeros of the derivative. Because the students had a conceptual understanding of the derivative, but did not know how to apply the rules for differentiation yet, they were forced to leave the derivation to the symbolic calculator.
Classroom observations showed that students managed to do so in a meaningful way. They

seemed to know what they were doing and no serious problems in interpreting the results were detected. One of them developed an efficient mixture between mathematical language and calculator language to write down the results (see figure 3).

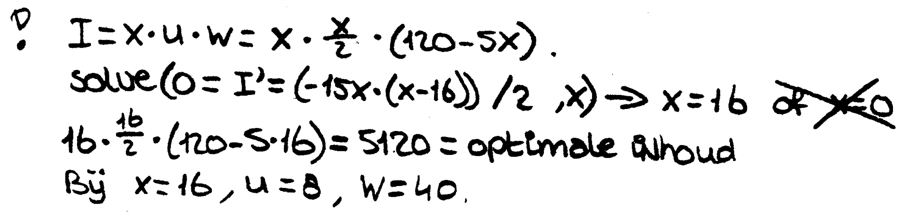


Figure 3: Student's notation of the solution procedure

The responses in the evaluative questionnaire point in a similar direction. One of the questions was: Could you save time using the symbolic calculator, because some operations can be done faster than manually? If yes, which are these operations? The 22 students unanimous replied 'yes'. Differentiation and solving equations were mentioned very often as time-saving procedures of the symbolic calculator.

Good news so far. During the experiment, however, some students found themselves not very happy leaving differentiation to the machine, while they did not know how the calculator 'did it' and were not able to check the results manually. Two illustrative quotations:

Esther solved a problem graphically.

Observer:	*It can also be done with differentiation.*
Esther:	*But I cannot differentiate this function yet.*
Observer:	*But the machine can.*
Esther:	*Yeah, but then you don't know what you're doing!*

Observer:	*(Talking to Melanie, one of the weaker students, who considers quitting the course) How are things going with the calculator, is it useful, or just an extra difficulty?*
Melanie:	*An extra difficulty. When you differentiate a function yourself, you know what you're doing.*

Similar reactions showed up in the written evaluation and the interviews after the experiment. One of the questions of the questionnaire was: What did you think of differentiating functions with the machine, that you cannot differentiate by hand? Out of 22 students, 7 replied 'not nice'. Typical motivations for this were:

- I have no idea what's happening.
- I want to see the logic of it, not just press buttons.

Two quotations from interviews after the experiment revealed the same uncomfortable feeling:

Interviewer:	*How was the experiment?*
Marieke:	*Nice, but not really useful.*
Interviewer:	*What do you mean by that?*
Marieke:	*You're typing in things blindly without knowing what you're doing exactly.*
Interviewer:	*Don't you trust the machine?*
Marieke:	*O yes, I suppose it does what you enter. I am just curious to know what's behind. I do want to use the calculator, but only when I could have done it myself as well.*

Diana:	*I felt uncertain while differentiating functions before it was explained. Wouldn't it have been better if you would have taught us the techniques first?*

The students knew that they were supposed to calculate the derivatives manually at their final examination. To some of them the main concepts in the unit were not the mathematization of real life optimization problems or the development of a problem solving strategy, but finding the derivatives. In this respect the experiment may have been asking too much of the students' patience to 'believe the machine'. Their teacher, however, considered this approach useful, because the students were really motivated to learn the rules of differentiation technique after the experiment.

The students' reactions make me think of the White-Box/Black-Box issue, that dominated the discussion about the pedagogy of computer algebra for some years. Buchberger (1990) suggested that computer algebra should only be used by students for tasks that they are able to perform by hand as well. Computer algebra is used as a black box, that could be opened by the students, if they would like to do so. Others (see Drijvers, 1995, for an overview) state that using the CAS for operations that are new to the students may elicit curiosity and can lead to interesting discoveries. The Black-Box/White-Box approach in this experiment seemed to work to a certain extent, but in the mean time elicited bad feelings among a minority, that consisted of relatively many girls and of weaker students (see also Zijlstra, 1999).
My interpretation is that at least some of the students want the 'boxes to be white'. I tend to take this tendency quite seriously, first because it indicates a good mathematical attitude, and second because the feelings that some students reported can frustrate the learning process. With the theory of RME in mind, I wonder whether the uncomfortable feeling of some students has to do with a lack of room for construction. The computer algebra device provides derivatives in a 'top-down' manner. Although it was not the aim of the unit, no environment is created for students to develop their own conception of the rules for differentiation. The process of vertical mathematization on this point is, at least temporarily, 'overruled' by the machine. After the experiment, when the students studied the rules for differentiation without the machine, this omission was 'repaired'.

6 Understanding algebraic concepts using a CAS

The second research question is: Can algebraic insight improve because of CAS use?
'Algebraic insight' is a broad concept. Therefore, I had to 'zoom in' at one specific aspect. In this study I confined algebraic insight to the ability to use the symbolic calculator for intentional rewriting a formula in a different, equivalent form and to notice the relationship between that particular form and a feature of the corresponding graph.

The unit contained a series of problems that ended with a 'free production task': students were encouraged to develop their own examples of functions that could be treated this way.

The following assignment was part of the test at the end of the experiment:

Consider the function y with $y(x) = \dfrac{6x^2 - 21x - 12}{2x^2 + 2x - 4}$.

Describe how you would re-write the formula so that the following feature can easily be seen:

a. the zeros;

b. the vertical asymptote;

c. the horizontal asymptote.

In figure 4 a typical student response to this assignment is shown.

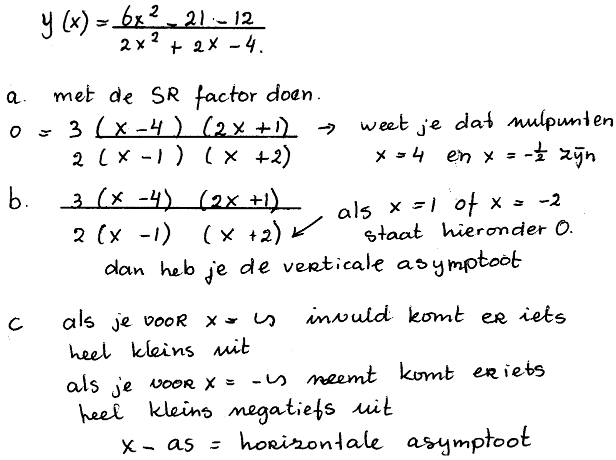


Figure 4: Esther’s answer

Esther made a mistake in copying the formula (see the first line of figure 4) but she entered the function correct in the machine. Then she applied the Factor-command, and noticed that the zeros can be found in the numerator of the result. For the vertical asymptote you can ‘read’ the zeros of the denominator, she explained.

For the horizontal asymptote she substitutes $x = \infty$. This does not work. Like most of her colleagues, she did not think of the Expand-command. Another option might have been to calculate $\lim_{x \to \infty} y(x)$.

Esther’s solutions are quite typical for the behaviour of many students: part a and b are solved, but c is often lacking. This was a bit disappointing. This may be due to the little attention that is paid to re-writing formulas after the students started to work on optimization.

7 Observations of student behaviour

This section contains five examples of student behaviour, that are typical and significant - at least that is how I perceive them - because they can help us in identifying obstacles that students experience while using a CAS.

7.1 Exact and decimal numbers

The left part of figure 5 presents the well-known weight problem, that Marquis de l’Hôpital described in 1696 in his book ‘Analyse des infiniments petits’. The central question is, given a length of 1 meter for the right rope: what is the lowest possible position of the weight? For a more detailed discussion of this problem see Drijvers (1996).

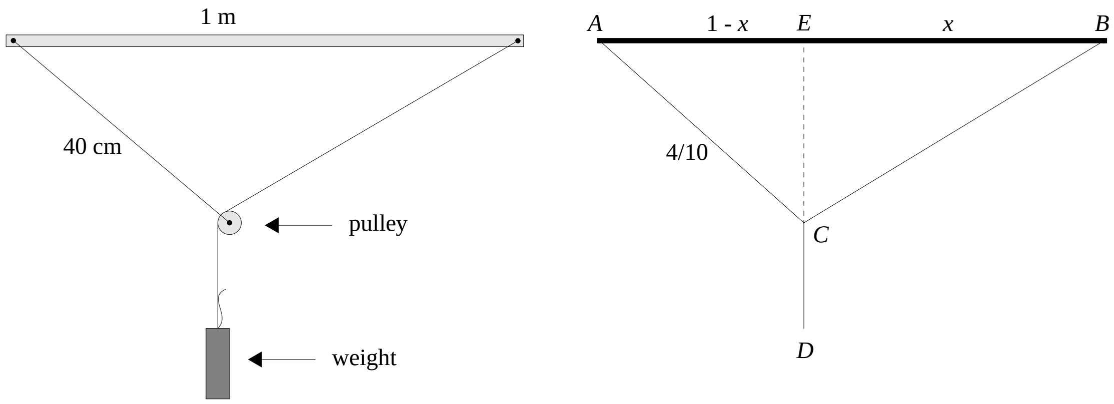


Figure 5: The weight problem of l'Hôpital

One of the students proceeds as follows:

$$
\begin{aligned}
EC &= \sqrt{0.4^2 - (1-x)^2} \\
BC &= \sqrt{x^2 + EC^2} \\
CD &= 1 - BC \\
ED &= EC + CD
\end{aligned}
$$

After entering these expressions as y1, y2, y3 and y4 respectively in the TI-92 in AUTO-mode, simplifying y4 yields the left screen in figure 6, whereas her neighbour gets the screen on the right. What is the matter?

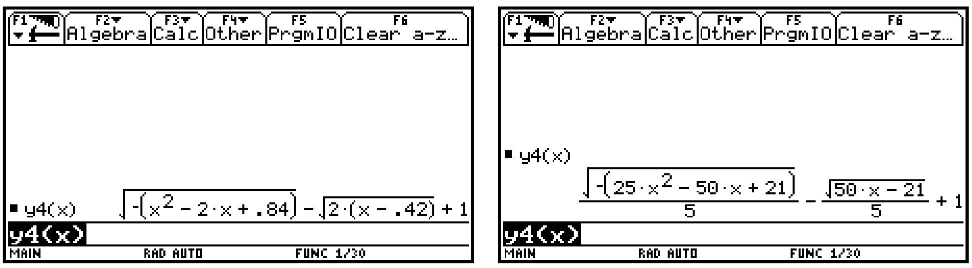


Figure 6: Why do the two TI-92 screens look different?

Apparently, her neighbour entered 4/10 instead of 0.4. Such different representations can also be caused by different calculation modes: approximate mode gives the left representation, whereas exact mode yields the right one. The differences between 4/10 and 0.4, and between exact and approximate mode are not clear to these students. This is an obstacle in understanding the different responses of the machine.

7.2 What is a simple algebraic representation?

In figure 7 you see a problem situation. A railway station *S* is to be situated on the railroad *CD*, so that the total distance from the station to the two cities *A* and *B* will be minimal. Where should this railway station be built?

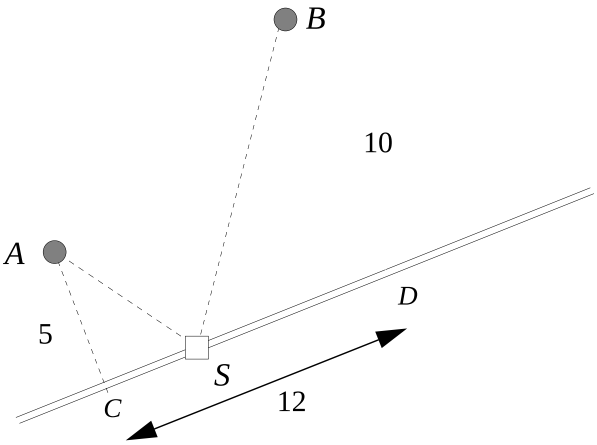


Figure 7: Where to build the railway station?

Students typically reacted like this:

$$DS = x$$

$$BS = \sqrt{10^2 + x^2}$$

$$AS = \sqrt{5^2 + (12 - x)^2}$$

The total distance is equal to $AS + BS$. Students type in the expressions and differentiate the total distance function with respect to x (see figure 8).

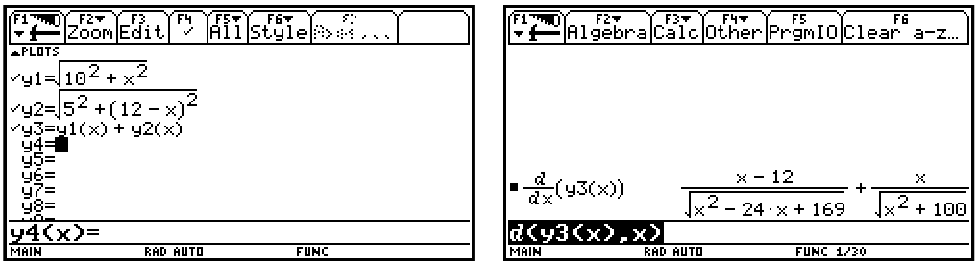


Figure 8: TI-92 screens for the railway station problem

The last part of the unit contains the solutions to the exercises. There, the left part of the derivative of y3 is represented as $-\dfrac{(12 - x)}{\sqrt{5^2 + (12 - x)^2}}$ in order to link the algebraic form to the ‘map’ in figure 7 (and maybe because this is the way ‘experts’ differentiate AS by hand). This leads to a student’s reaction: “In the solutions, there’s a minus in front!”. She clearly had difficulties in understanding that the two formulas are equivalent. Of course, the denominator looks different as well. The way the CAS represents the solution can be different from the representation that the user considers as the most simple in a specific situation.

7.3 Helping the machine

Continuing the problem of the railway station in the previous section, students wanted to calculate the zeros of the derivative. The ‘old’ TI-92 that these students had - without ‘Plus-module’ - solves the equation in AUTO-mode to $x = 8$. (see left screen of figure 9). Here the students usually did not notice the point behind the 8, indicating an approximate result. In EXACT-mode, the machine returns an empty solution set. (At present, the TI-92 can solve this equation in exact mode.) Again, the students encounter the exact-approximate obstacle that I described in section 7.1.

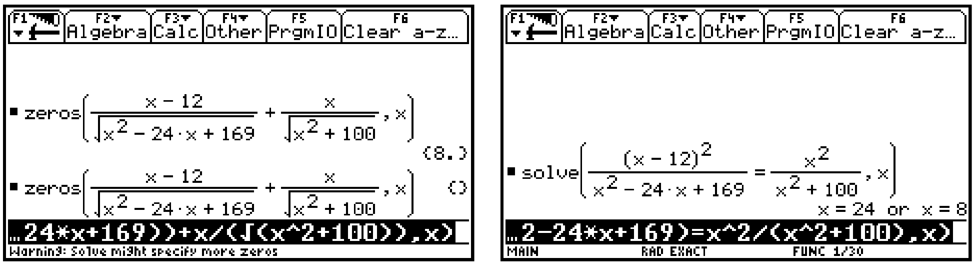


Figure 9: Helping the machine to find an exact solution

In order to ‘help the machine’, it was suggested to square the two parts of the derivative. As you see in the right screen of figure 9, this yields two exact solutions, one of which is due to the squaring. Finding out how to ‘help the machine’ to overcome its limitations is not an easy thing for students.

7.4 What can computer algebra do for you?

Two observations illustrate that students are not always aware of the algebraic power of the tool that they have in their hands. Taking advantage of that is not as obvious as it may seem.

In the first investigation task, faculty numbers $x!$ are considered. The question is how one can find out how many zeros there are at the end of $x!$ without having to calculate $x!$ itself. This example is described in detail by Trouche (1998).
In the first observation two boys are developing a (beautiful!) procedure that will calculate the number of zeros at the end of $x!$. They discovered that one has to divide x by the subsequent powers of 5, and then add up the integer parts of the outcomes. They entered $y1(x) = \sum_{k=1} int\left(\frac{x}{5^k}\right)$ but they did not know what to fill in as the upper boundary of the summation. First, they tried infinity, “certainly enough”, as they said. Unfortunately, the TI-92 did not accept this. Then they took 20 as upper limit, which worked “as long as x is not too big”. Too big, they realized, means exceeding 5^{20}. Thinking about this, they found out that the upper bound should be the biggest n, so that 5^n does not exceed x. However, they were unable to solve $5^n = x$ for n by hand. They tried $x^{1/5}$, and this seemed to work, although this was not the solution of the equation, as they knew. The point here is, that in spite of (or maybe because of?) the quite sophisticated work they were doing, they did not realize that the TI-92 would easily solve this equation for them!

The second example is this section is somewhat easier.
A sheet of paper - originally 21x29.7 cm A4-format, later of dimensions a and b - is folded so that the upper left corner touches the ‘base’ (see figure 10). The question is to calculate the maximum surface of the shaded triangle.

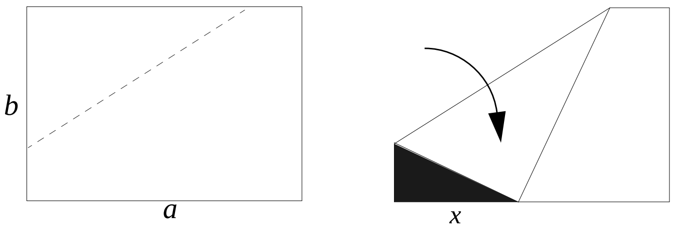


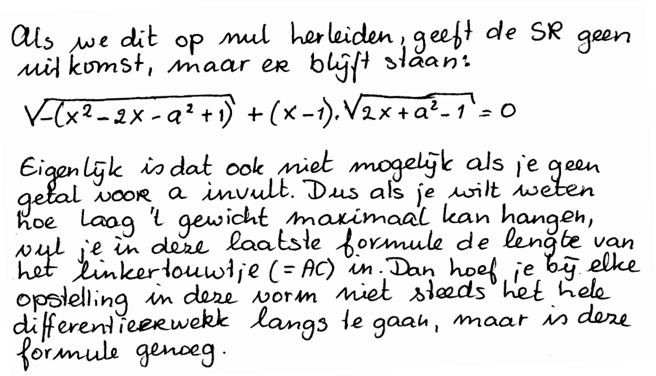
Figure 10: Folding a sheet of paper

Students usually called the height of the triangle *y*, and obtained: $\sqrt{x^2+y^2} = b-y$. They then wanted to isolate *y*, but did not seem to use the symbolic calculator for that. Again, they did not take advantage of the power of the tool. (By the way, the triangle with the optimal surface happens to be half of an equilateral triangle. Why would that be?)

This difficulty to use the symbolic calculator for algebraic manipulations may have to do with the concepts of variables and parameters, that show up in the next section.

7.5 Parameters and variables

Two examples indicate that some students had problems in understanding the role of parameters. Let us first look back at the weight problem of l'Hôpital (see section 7.1). In the end, the length of the left rope, originally 40 cm, is generalized to a parameter *a*. The derivative of the height-function in this case equals zero when $\sqrt{2x-x^2+a^2-1}+(x-1)\sqrt{2x+a^2-1} = 0$. The students' TI-92 did not solve this equation. (Again, the TI-92 Plus does...) In fact, this could have been an example in section 7.3 of a situation in which this machine needs some help. But look at a student's reaction to this (see figure 11).


Als we dit op nul herleiden, geeft de SR geen uitkomst, maar er blijft staan:

$\sqrt{-(x^2-2x-a^2+1)} + (x-1).\sqrt{2x+a^2-1} = 0$

Eigenlijk is dat ook niet mogelijk als je geen getal voor a invult. Dus als je wilt weten hoe laag 't gewicht maximaal kan hangen, vul je in deze laatste formule de lengte van het linkertouwtje (= AC) in. Dan hoef je bij elke opstelling in deze vorm niet steeds het hele differentieerwerk langs te gaan, maar is deze formule genoeg.

Figure 11: Irene's explanation for finding no solution

Let me translate the relevant part of this:

> *When we put this to zero, the symbolic calculator gives no solution, but it returns (equation) In fact, this (a solution, PD) is not possible when one does not substitute a value for a. (...)*

The second example is a well-known problem: a pipe is to be carried horizontally around the corner of a corridor. The question is: how long can the pipe be? After concrete dimensions were given, the situation is generalized to corridors of dimensions *p* and *q* meter (see the left part of figure 12).

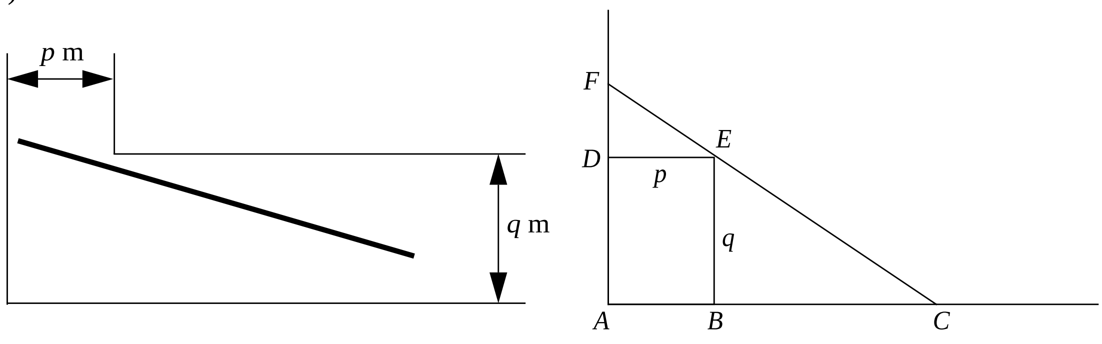


Figure 12: The pipe in the corridor

The final question is to express the maximal length of the pipe that can pass in terms of p and q. Dennis and Niels started like this (see the right part of figure 12):

$$BC = x$$

$$CE = \sqrt{x^2 + q^2}$$

Then they realized that triangle *CBE* is similar to *CAF*. The factor of multiplication is $\frac{x+p}{x}$,

so $$CF = \frac{x+p}{x}\sqrt{x^2+q^2}.$$

They did not, however, work this out further. During the classroom discussion afterwards, I understood why:

The teacher asks for solutions of the equation.
Dennis: *You first have to fill in values for p and q, don't you?*
Teacher: *You can also solve immediately. Then you get the answer expressed in p and q.*

To them, letters need to have values before one can proceed, or at least before one can apply the solve-command. 'Solving' to them probably stands for 'finding a numerical result'. Expressing one variable in terms of the other apparently is not really 'solving'.

8 Obstacles using a CAS

Reading section 7, one might argue that the reported observations are just incidents of students' errors. As Treffers (1993) describes, however, seemingly incidental errors can guide the researcher to new insights. Let me quote Wenger here, when he explains the debuggy metaphor (Wenger, 1987, p. 222):

> *The essence of the debuggy metaphor is that there are identifiable and systematic underlying causes of students' errant behaviour and that these causes often have deep and pedagogically powerful implications.*

The observations led to the identification of some obstacles that students encounter when using a CAS, which was the third and last research question. I will briefly discuss them below. The numbers correspond to the subsections of section 7.

1. *The difference between numerical and algebraic calculations.*
   Sometimes a CAS performs exact, algebraic calculations and sometimes it does approximated calculations. A single button can thus evoke two conceptually different methods. In order to understand the 'status' of a result, students should be conscious of this. They should be able to classify the CAS output in this respect, and know how to influence this and how to choose between the two approaches.

2. *The CAS provides algebraic representations that are different to what students expect and conceive as 'simple'.*
   The computer algebra routines have their own rules for simplifying expressions, that may not result in what the user considers to be the most simple representation in a specific situation. Students need to cope with that. They should know, for example, how to check whether two expressions are equivalent or not. Furthermore, they should develop 'an eye' for the way a CAS operates while determining the algebraic representation, so that some representations become 'logical' to the student.

3. *The CAS has its limitations and it may need some help to overcome them*.
   Students often do not know what to do when the symbolic calculator does not give an answer. They do not have the feeling that they might try to help the machine by specifying a domain, by squaring to get rid of roots, by choosing another precision mode or so. For optimal help, the user should have an idea of why the machine does not do what it is meant for, and of a step that might be in the right direction. This requires an understanding of algebraic strategies.

4. *The ability to decide when and how computer algebra can be useful*.
   Students sometimes do not realize how computer algebra can help them. The observations provided examples of not using the solve-command to isolate a specific variable. In order to improve this, students have to develop a clear view of what they can expect from the CAS. Maybe this requires a thorough familiarity with the algebraic potential of the tool, that the students in the observations did not have yet. In fact, after having used the graphing calculator for more than a year, this machine has become an integrated part of (school)-life to many of the students, whereas the symbolic calculator in this short period did not. Computer algebra may be a more complex phenomenon than a graphing calculator.

5. *Using the CAS requires good insight into variables and parameters*.
   As Usiskin (1988) pointed out, letters in a CAS are not placeholders for numbers, but just symbols. For the user, this often is not the case. Adequately operating with symbols using computer algebra requires that students are aware of this, and that they really understand the concept of variables and parameters. Managing a CAS probably requires that the algebraic insight of the students is at the ‘symbolic level’ (see Harper, 1987).
   Is the computer algebra environment just making the difficulties with variables and parameters more explicit? Or is this inherent to the use of symbolic manipulation software, that variables and parameters come along in a more abstract context, that enlarges possible misconceptions?

9 Concluding discussion

Looking back at this short classroom experiment, I will summarize the conclusions with respect to the three research questions. Then I will briefly consider the role of the theoretical framework.

As far as the resequencing of concepts and skills is concerned, the results are ambiguous. On the one hand, students showed understanding of the concepts of mathematizing optimization problems and of the strategy of solving them. On the other hand, some of them felt uncomfortable about leaving the applications of the rules of differentiation to the symbolic calculator. The theory of RME provides a possible explanation of these feelings: some students probably experience a lack of construction room to reinvent the rules of differentiation, and were reluctant to follow this ‘top-down’ approach. Although rediscovering of differentiation rules was not the aim of the instructional unit, it is my conviction that such feelings have to be taken seriously, because they frustrate the learning process.

Improvement of algebraic insight by means of using a CAS is an interesting option. The idea of establishing the relation between algebraic form and graphical feature was only partially successful. I conjecture that this is mainly due to the educational setting, where this aspect did not get enough attention. More generally speaking, developing algebraic insight at upper secondary level is a hard issue; the main topics are usually functions and calculus, and whereas algebraic skills are required there, algebra in itself usually is not on the agenda, at least in the Netherlands.

Identification of obstacles using computer algebra, however, reveal that there may be other algebraic concepts involved, that obstruct optimal CAS use. The examples in section 7.4 and their interpretation in section 7.5 illustrate the relevance of a good understanding of variables and parameters, and a notion of how a CAS deals with them. This might be an interesting point to be worked out in detail in future.

Much has been said about the importance of integrating the process character and object character of mathematical concepts (see Sfard, 1991 and Gray&Tall, 1993). Monaghan (1994) indicated that in the CAS environment functions and formulas are considered as objects, whereas the processes become invisible. I wonder whether the structural way in which computer algebra deals with functions and variables as objects may enlarge the conceptual difficulties that are described above.

More generally speaking, I think that the relevance of the identified obstacles is not restricted to the specific platform of the TI-92, but that these factors have to be taken into account whenever students learn to use computer algebra. As indicated above, the difficulties are encountered when using a CAS, but they have links with the understanding of mathematical concepts. As a pedagogical strategy I would advise to consider them seriously, to pay attention to them and to take advantage of them by making explicit the mathematics behind them. Trying to avoid them might be a bad pedagogical strategy, I believe.

Let me now look back at the theoretical framework. The theory of RME clearly guided the development of the instructional units. After the experiment, the difficulties reported with the resequencing could be explained in terms of the theory. The externally set research questions, however, restricted the role of RME in this study. The beliefs that CAS can support the students' flexibility and that a CAS environment can support the reinvention of mathematical concepts are elaborated only partially.

More attention has been paid to the identification of obstacles, which in itself is a useful step in considering the pedagogical use of the tool. The developmental research design typically makes use of the qualitative, close-to-the-students observations that facilitate this identification. Another characteristic of developmental research is that students' erroneous behaviour is seen as a source for further development of the theory and the educational experiment. In this case time was lacking for a real cyclic process, in which the findings were used as 'feed-forward' during the experiment. In a follow-up study, however, they will be serve as a starting point.

In spite of the factors that limited the role of the theoretical framework, the study presented here shows how the theory of RME and Developmental Research guide the research on the integration of computer algebra in mathematics education.

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FORUM 3/REACTION
Understanding algebra and using CAS. A reaction to "Students encountering obstacles using CAS: A developmental-research study"

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Introduction

First, on behalf of us all, I wish to thank Paul Drijvers for providing an excellent article to stimulate our discussion of the place of CAS in school mathematics. Within a framework of Realistic Mathematics Education, Drijvers has described an educational experiment to explore the effects of using CAS with senior secondary students. His paper centres around three questions:

1. Is it possible to resequence a course using CAS, so that concept development precedes the solving techniques and algorithms?
2. Can algebraic insight improve because of CAS use?
3. What obstacles do students experience while working with computer algebra?

In answering these three questions, Drijvers provides us with several rich reports of situations and student behaviours that throw up many other interesting propositions. These questions are important for creating policies about CAS in schools and learning how to teach with it.

In discussing these questions, I will draw on experiences from experiments that we have carried out at the University of Melbourne. One experiment by Barry McCrae, Gary Asp and Margaret Kendal (1999) took place over 10 weeks in three Year 11 classrooms with 59 students doing the first introduction to calculus. The investigators wanted to use CAS to emphasise conceptual understanding rather than learning the calculus rules. Robyn Pierce has been teaching several tertiary courses of introductory calculus using CAS (Maple and more recently Derive) for several years. In 1998, the reactions of her students to the use of CAS were systematically monitored and the results are reported in Pierce (1999). David Tynan and Gary Asp (1998) worked with two classes of Year 9 students (48 students at one school), comparing a program on linear equation solving and simple polynomial factorisation using the CAS calculator TI-92 with a program using the graphics calculator TI-82.

Question 1. Can courses be resequenced using CAS so that concept development precedes solving techniques and algorithms?

The possibility of a new tool permitting resequencing the teaching of calculus has been explored with positive results in many studies (Heid, 1997), as in Drijvers' study. We all hope and expect that leaving the rules for differentiation until last might shift the primary emphasis in introductory calculus from the rules to the concepts and applications. Drijvers outlines two carefully constructed units, which feature imaginative problems requiring optimisation in real life situations. The primary new concept to be taught was the link between extreme values of a function and the zeros of the derivative. Although little detail is given in the report, the students apparently easily grasped the concepts underlying optimisation and were able to carry out the procedures with the CAS and interpret the results. This study therefore provides us with one example of what new resequenced curricula may look like: when the concept of derivative is in place, CAS can be used to enable students to see the uses of the derivative, before extensive skill in differentiating is acquired.

Working with younger students beginning algebra, Tynan and Asp (1998) also tried to resequence in a way that closely paralleled Drijvers' use. They used CAS to show students the power of algebraic methods of solving problems, before the students could reliably solve one variable linear equations by hand. All too often, students learn how to solve equations, but when faced with a word problem or a problem situation they do not realise that equation solving is relevant and instead resort to numerical methods (Kieran, 1992). Tynan and Asp (1998) reported clear success in getting students to appreciate the power of algebraic methods by implementing a curriculum which began with setting up equations, and solving them first with the CAS, before by-hand procedures were in place. For example, on one item, 56% of student taught with CAS set up an equation and tried to solve it using algebraic "do the same to both sides" methods. In the class taught traditionally, only 26% did this.

This use of CAS to reorder skills and problems is one possibility for reorganising the curriculum, but there are others. For example, instead of spending the time which is freed by less emphasis of skills on encountering more demanding and interesting problems (as did Drijvers), McCrae, Asp and Kendal (1999) decided the first priority was to use this time to build conceptual understanding very carefully. Their students were therefore given 10 lessons focussing on the concept of a derivative before beginning to differentiate, first by hand from first principles and then with CAS. The study found that the results of the Year 11 students were comparable with the results of the Year 12 students (one year older and about to undertake the final school examinations), indicating that the students in the experimental program had indeed made excellent progress. In the initial concept building, however, graphical and physical (datalogging) tools were more useful than the symbolic capability of CAS. In the discussion session, we may be able to discuss the reasons for this.

Despite their success, McCrae et al found that some concepts were not understood as well as had been expected and they found instances where using CAS may mask a lack of understanding. For example, with CAS available, 32 students chose to differentiate the function $1/x^6$ with CAS and all but one were correct. However, without CAS, only 10 of these students were successful and 13 gave the answer x^{-6}. (It had not been regularly practised in the course). These students had appeared to understand the derivative and its uses quite well. However, differentiating with CAS required only one step and we think that quite possibly they also thought that differentiating by hand will also take only one step. The one step of the by-hand procedure that they carried out was the preliminary step of finding an equivalent algebraic form. Although we do not have conclusive evidence on this point, we believe that did not know the differentiation was unfinished.

Why don't all students like CAS?

Drijvers' own conclusion about the advisability of resequencing is qualified – students learned well but about a third of them did not feel comfortable with it. His recommendation is that curriculum developers should take this feeling of unease seriously. I found this admonition one of the most thought-provoking aspects of the paper.

It is easy to be dismissive of students' uneasiness, and I have done this in the past. One might implicitly believe, for example, that students are enculturated into an impoverished mathematics so that they see the underlying ideas as unimportant and the rules and procedures as of utmost importance. The feelings of uneasiness are therefore attributed to the current bad situation. Alternatively, one might dismiss their feelings of uneasiness because they result (for pragmatic reasons) from having an unreformed assessment system, where CAS cannot be used. In Drijvers' study, for example, the students had to calculate derivatives manually at their final examination. This factor also applied in the McCrae et al (1999) study, where 15 of the 59 students (a similar proportion to Drijvers one third) made at least one written comment during the study which expressed some uneasiness or concern, including these below:

Pupil 45: *"The TI-92 was great in working out the answers quickly whilst doing the exercises, but it was no good once we had to do the class test. We should have used the time we spent on the calculators practising the formulas."* [Note: Students did class tests with and without CAS and will do the final examination in the following year without.]

Pupil 21: [The TI-92 did not help me learn] *"how to do things with 100% confidence by myself"*.

Pupil 50: (Commenting on things that she had found frustrating) *"If it showed how it was working things out – so I understood all the steps instead of just finding the right answers but not understanding how I got there."*

Pierce (1999) working with tertiary students also finds this uneasiness, although her students have unconstrained access to Derive in the examinations. For example, 13 students undertaking an introductory calculus unit were asked if they thought CAS would offer fresh hope to students with difficulties in mathematics. In their written responses, 6 expressed some sort of uneasiness including:

Student 6: *"It is a good idea, however although the use of computers will give you the correct answer (as long as entered correctly) it does not let the user know what they have done or how they would go about getting the answer [….] manually"*.

Student 7: *"No, I think you need to know what the computer is working out and how"*.

Student 8: *"CAS offers the student the answers without really needing the basics. I personally would have enjoyed more time spent on the basics."*

Pierce's work leads us to look for other explanations: student's uneasiness is deeper than concern about the examination requirements.

Drijvers' own explanation is that "some students probably experience a lack of construction room to reinvent the rules of differentiation". The students learned in a previous unit how to differentiate powers of x (presumably thereby constructing differentiation as a concept and process for themselves). In the unit being studied, they were using the CAS to differentiate expressions such as the square root of (x^2 + 100). Is it likely that they need to reinvent for themselves the chain rule? I think not. It seems not to be the underlying concepts, but the rules to differentiate complex combinations that are causing the uneasiness. Perhaps the students do not know that there are rules for finding the derivative of complex combinations of simple functions from the derivatives of the simple functions themselves. Possibly calculating the square root of a number on a scientific calculator is a transparent operation because students can imagine what the calculator may be doing (e.g. trying various numbers) even if it is not doing this at all, whereas they may not be able to imagine how a CAS can differentiate. The feelings of uneasiness may disappear if they are told a little about how the CAS works, in the simplest of terms. For students, the black box may be much blacker than we think. A profitable avenue for research might be to explore from the students' point of view what makes a box appear "blacker" or "whiter".

Question 2. Can algebraic insight improve because of CAS?

Drijvers' limited his comments on the improvement of algebraic insight from the use of CAS to being able to rewrite an expression to highlight certain features of a graph. He found some success on this, but not as much as desired. However, several other situations reported in the study relate to algebraic insight, with a broader definition, and particularly to the level of algebraic insight required for CAS use.

Firstly, there are instances reported where CAS provides algebraic representations that are different to what students expect and conceive as "simple". Dealing with unexpected output certainly requires algebraic insight. One example that our group has recently discussed is the solution of the problem in the diagram below. This is a sensible, mainstream application of trigonometry – the sort of problem that we hope may become more accessible with CAS. With the length of D or H given numerically, it is a straightforward problem, substituting in the three basic relationships:

(1) $\tan A = \dfrac{H}{D}$

(2) $\tan B = \dfrac{H}{d+D}$

(3) $\tan C = \dfrac{h+H}{d+D}$

Without numeric values, it requires moderate algebraic skill to find the solution:

$$h = \frac{d \tan A(\tan C - \tan B)}{(\tan A - \tan B)}$$

This is (almost) the solution given by Mathematica (first eliminate H and D and then solve for h) and by Maple V (solve command). However, to use the solve command on the three equations with Derive and the Texas Instruments calculators (TI-92 and TI-89), the equations first have to be transformed by hand to get them into a suitable format. The CAS itself cannot be used to do the required rearrangement, because symbols will be put in the wrong order (e.g. D * tanA rather than tanA * D). The solve command then gives the unexpectedly complicated solution below, which is made much worse by the fact that it is too big to fit on the TI-89 screen.

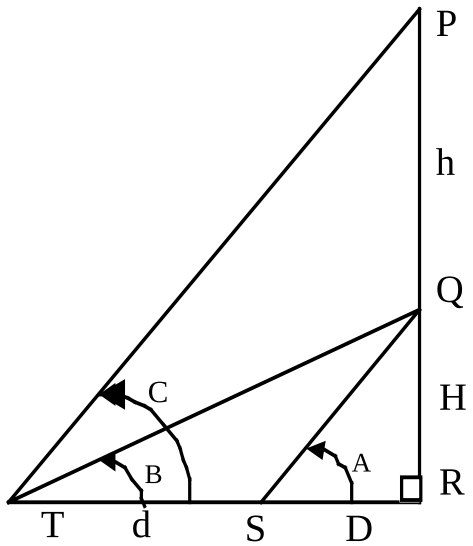
$$h = \frac{-\sin A(\cos B \sin C - \sin B \cos C)d}{(\cos A \sin B - \sin A \cos B)\cos C}$$

This is a straightforward problem, but it does not get a straightforward answer. This problem illustrates that students need a great deal of algebraic insight to use CAS. It also forewarns of the difficulties of assessing mathematics fairly with CAS: students using different machines will encounter different obstacles

Beyond the constrained world of school mathematics exercises, it is common in mathematics that simple questions may have non-simple answers. Sometimes, as in the case, above, the lack of simplicity is just length. At other times, the lack of simplicity is profound. Many quadratics with simple coefficients, for example, have simple answers, but some (such as $x^2 + 1 = 0$) open up a whole new branch of mathematics. A major challenge for the development of algebraic insight when using CAS is for students to appreciate when they simply have a cumbersome answer and when the answer contains something quite new for them.

An aside

Using a new tool causes us to see mathematics differently (Tynan et al, 1995). Because I was thinking how to find *h*, given the relationships (1), (2) and (3) for the first time with a CAS, I came to see these constraints simply as three linear equations in three unknowns (H, D and *h*), so the problem could be solved using the matrix linear algebra facilities. Powerful tools provide many more feasible solution paths than we are used to.



A flagpole (PQ in the diagram) is placed on top of a castle wall (QR), which is surrounded by a moat (RS). From point S, the angle of elevation of the top of the wall is A degrees. From point T, the angle of elevation of the top of the castle wall is B degrees. From point T, the angle of elevation of the top of the flagpole is C degrees. The distance ST is *d* metres. Find the height of the flag pole in terms of A, B, C and *d*.

Question 3. What obstacles do students experience with CAS?

The article lists six obstacles that Drijvers' students experienced when using CAS. We have also found evidence of similar obstacles and others. On one occasion in McCrae, Asp and Kendal's study, a great deal of lesson time was lost because of the difference between approximate and exact calculations and fixed students' expectations. Everyone in the class factorised $2x^2 + 3x - 2$ but there seemed to be two different answers:

$(x+2)(2x-1)$ and $2(x-5)(x+2)$

The teacher was at a loss to explain – perhaps the CAS was not working correctly, or perhaps wrong commands had been used. As these explanations and others were investigated fruitlessly, the clock ticked away, wasting valuable lesson time. Finally, someone saw the decimal points: the CAS had not given the second factorisation above but had correctly given

$$2.(x-.5)(x+2.)$$

Everyone had missed seeing the decimal points because they had not been expected in this circumstance. On the test at the end of the unit, similar mistakes were made.

Is CAS too sensitive to failure?

Incidents such as the above suggest that today's CAS is still too sensitive to failure for schools to find it extremely useful and that it may well remain marginal (despite impending price drops) until it enters a new era in user-friendliness. The difficulty of getting the syntax right plagues students. Reading the output on the price-accessible CAS calculators is too hard. The error messages are simply not smart enough. When faced with an equation that it is unable to solve in exact mode (as was the CAS in Drijvers' study with an equation with too many square roots (section 7.3)), the CAS should send an informative error message rather than an empty solution set. A helpful message could be: "CAS cannot find the zeros – try one of the following" followed by a list suggesting changing to approximate or auto mode, rewriting the equation in another form, or other possibilities. Perhaps the CAS could give a message about the number of roots of an equation asking the user to be alert when apparently too many or too few are given. CAS may be more useful in school when it begins to share, rather than simply use, deep expert knowledge.

The functional and pedagogical uses of CAS

Etlinger wrote an article in 1974 about the introduction of electronic calculators in schools that is relevant today for the introduction of CAS. For example, he describes the spectrum of possible uses of calculators in schools ranging from strictly functional uses (as a tool to get answers, replacing by-hand techniques just as clocks and watches have replaced telling the time by shadows) to strictly pedagogical uses (to facilitate learning within an unaltered curriculum). He asks if the calculator will retain its motivational value if it becomes a household object (sadly not, we now know!) and whether children will come to understand the concepts behind arithmetic operations better or not so well when they are mediated by a black box. He wonders if calculators will help or hinder learning by hand skills and whether children will think more or less about the methods of solving a problem.

Even at that early stage, Etlinger noted that "in many cases it is the limitations of the calculators which are directly responsible for the pedagogical value of the machine" (p. 44). One of his illustrations was to motivate the study of iterative algorithms by using a calculator to iteratively calculate square roots, a facility not standard on 1974 machines. In recent years, we have also seen the limitations of graphics calculators becoming a feature of their pedagogical use. For example, in *Graphic Algebra* (Asp, Dowsey, Stacey and Tynan, 1998) we frequently exploited the fact that the default viewing window did not show the salient features of a graph. This was a useful pedagogical technique to improve students' understanding of the shape of graphs and concomitantly a useful assessment technique, but it will be less useful in the future as new models are given some capacity to select an appropriate viewing window. In each case, the machine has enormous power (multiplying, drawing graphs), but the pedagogical opportunity often arises at the edge of this power.

Will the same situation apply with CAS? Do the limitations of CAS increase or decrease its pedagogical use? Most of the examples given in Drijvers' study point to the limitations of CAS decreasing its pedagogical use – the difficulties of getting the right mode, of recognizing different forms of a right answer etc. Instead of stimulating good discussion, these limitations seem to have become points of frustration in Drijvers' study and also in our own. Drijvers noted that "students [were] not always aware of the algebraic power of the tool that they have in their hands". (section 7.4). It is easy for students to underestimate the power of CAS. In the same way, it is easy for teachers and researchers to underestimate the very marked changes in thinking that students need to undergo as they come to understand the branches of mathematics that are embedded in even the simple CAS systems. Paul Drijvers' paper illustrates how developmental research is needed to uncover excellent ways to use CAS to enhance the outcomes of mathematical education.

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